

Recognizing when closed braids admit a destabilization, an exchange move or an elementary flype

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Dedicated to Joan Birman on the occasion of her advancement to active retirement.

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Abstract

The Markov Theorem Without Stabilization (MTWS) established the existence of a calculus of braid isotopies that can be used to move between closed braid representatives of a given oriented link type without having to increase the braid index by stabilization [BM4]. Although the calculus is extensive there are three key isotopies that were identified and analyzed—destabilization, exchange moves and elementary braid preserving flypes. One of the critical open problems left in the wake of the MTWS is the *recognition problem*—determining when a given closed n -braid admits a specified move of the calculus. In this note we give an algorithmic solution to the recognition problem for these three key isotopies of the MTWS calculus. The algorithm is “directed” by a complexity measure that can be *monotonically simplified* by the application of *elementary moves*.

1 Introduction.

1.1 Preliminaries.

Given an oriented link $X \subset S^3$ it is a classical result of Alexander [A] that X can be represented as a closed n -braid. For expository purposes it is convenient to translate Alexander’s result into the following setting. Let $S^3 = \mathbb{R}^3 \cup \{\infty\}$ and give \mathbb{R}^3 an open-book decomposition, i.e. $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ is fibered by a collection of half-plane fibers $\mathbf{H} = \{H_\theta | \theta \in [0, 2\pi]/0 \sim 2\pi (= S^1)\}$ where $\mathbf{A} = \partial \tilde{H}_\theta$ is the z -axis. Equivalently, we consider the cylindrical coordinates (r, θ, z) on \mathbb{R}^3 and H_θ is the set of points in \mathbb{R}^3 having their second coordinate the fixed $\theta \in S^1$.

An oriented link X in $\mathbb{R}^3(\subset S^3)$ is a *closed n -braid* if $X \subset \mathbb{R}^3 \setminus \{z\text{-axis}\}$ such that X

transversely intersects each fiber of \mathbf{H} in n points. The *braid index* of X is the cardinality $b(X) = |X \cap H_\theta| = n$ which is invariant for all $H_\theta \in \mathbf{H}$.

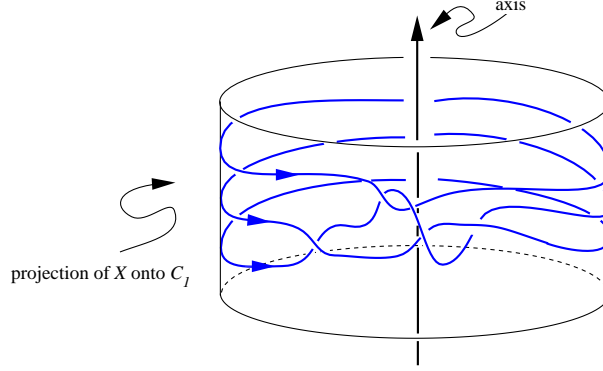


Figure 1:

Possibly after a small isotopy of X in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$, we can consider a regular projection $\pi : X \rightarrow C_1$ given by $\pi : (r, \theta, z) \mapsto (1, \theta, z)$, where $C_1 = \{(r, \theta, z) | r = 1\}$. The projection $\pi(X) \subset C_1$ is isotopic to a *standard projection* that is constructed as follows. For $n = b(X)$ we first consider the circles: $c_i = \{(r, \theta, z) | r = 1, z = \frac{i}{n}\}$, $1 \leq i \leq n$. Next, we alter the projection of this trivial unlink of n components to construct the projection of $\pi(X)$ by having adjacent circles c_i and c_{i+1} cross via the addition of a positive or negative crossing at the needed angle to produce a projection of X . It is clear through an ambient isotopy of \mathbb{R}^3 which preserves the fibers of \mathbf{H} that we can always reposition X in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ so that $\pi(X)$ is a standard projection. (See Figures 1 and 2.)

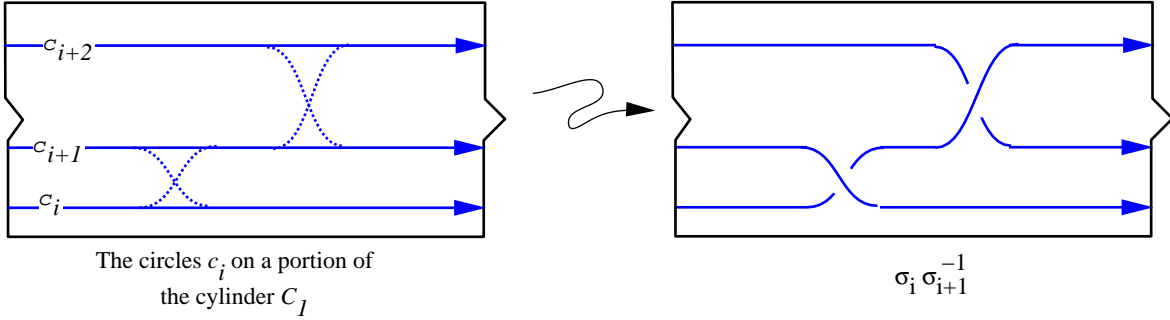


Figure 2:

From a standard projection $\pi(X)$ on C_1 we can read off a cyclic word in the classical Artin generators $\sigma_i^{\pm 1}$, $1 \leq i \leq n-1$. Specifically, $\beta(X)$ is the cyclic word that comes from recording the angular occurrence of crossings where a positive (respectively negative) crossing between the circles c_i and c_{i+1} contributes a σ_i (respectively σ_i^{-1}) element to $\beta(X)$.

We can alter $\pi(X)$ (and, correspondingly, $\beta(X)$) while preserving the n -braid structure of

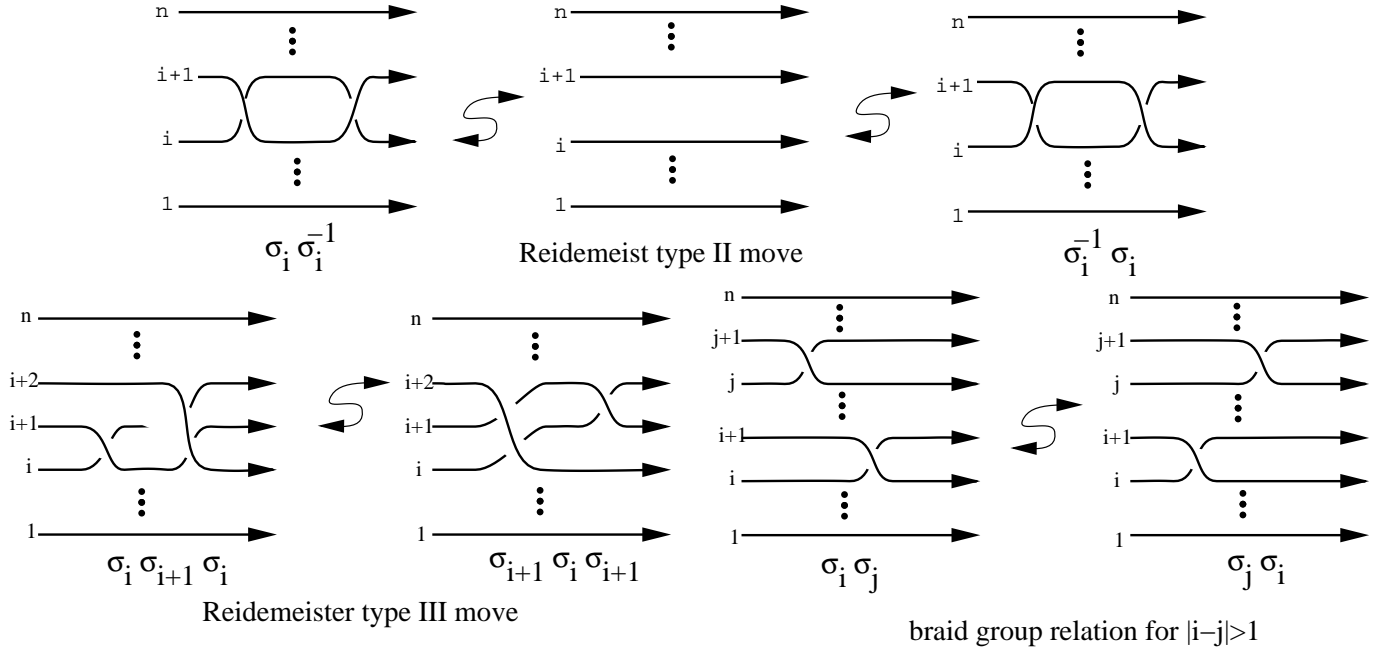


Figure 3: Generally, type III moves correspond to $\sigma_i^\epsilon \sigma_{i+1}^\delta \sigma_i^\delta$ interchanged with $\sigma_{i+1}^\delta \sigma_i^\delta \sigma_{i+1}^\epsilon$ where $\epsilon, \delta \in \{-1, 1\}$. Generally, the braid group relation is $\sigma_i^\epsilon \sigma_j^\delta$ interchanged with $\sigma_j^\delta \sigma_i^\epsilon$ where again $\epsilon, \delta \in \{-1, 1\}$.

X , plus its standard projection characteristic, by use of type-II and -III Reidemeister moves and the braid group relation. (See Figure 3.)

We consider the equivalence classes under the moves in Figure 3. Specifically, X and X' are *braid isotopic* if $\pi(X)$ can be altered to produce $\pi(X')$ through a sequence of type-II and -III moves, and braid group relations. We will let $\mathcal{B}_n(X)$ be notation for the equivalence class of n -braids which are braid isotopic to X .

Next, let \mathcal{W}^t be all words generated by the set $\{\sigma_1^{\pm 1}, \dots, \sigma_t^{\pm 1}\}$ and \mathcal{U}^s be all words generated by the set $\{\sigma_s^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}\}$. Initially we allow for the possibilities of $\mathcal{W}^t \cap \mathcal{U}^s = \emptyset$ or $\mathcal{W}^t \cap \mathcal{U}^s \neq \emptyset$, but exclude $\mathcal{W}^t = \mathcal{U}^s$. We now have a sequence of definitions.

An n -braid X admits a *destabilization* if for $\pi(X)$, its associated braid word $\beta(X)$ is of the form $W\sigma_{n-1}^{\pm 1}$, where $W \in \mathcal{W}^{n-2}$. (See Figure 4(a).) The $(n-1)$ -braid Y with $\beta(Y) = W$ (and corresponding projection $\pi(Y)$) is obtained by a *destabilization* of X .

Consistent with the terminology in [M1], an n -braid X admits a *double destabilization* for if for $\pi(X)$, its associated braid word $\beta(X)$ is of the form

$$W\mathbf{X}^{\epsilon, 2} = W\sigma_{n-2}^\epsilon \sigma_{n-1}^\epsilon \sigma_{n-3}^\epsilon \sigma_{n-2}^\epsilon,$$

where $W \in \mathcal{W}^{n-3}$ and $\epsilon \in \{-1, +1\}$. (Thus, $\mathbf{X}^{\epsilon, 2}$ is a positive ($\epsilon = +1$) or negative ($\epsilon = -1$)

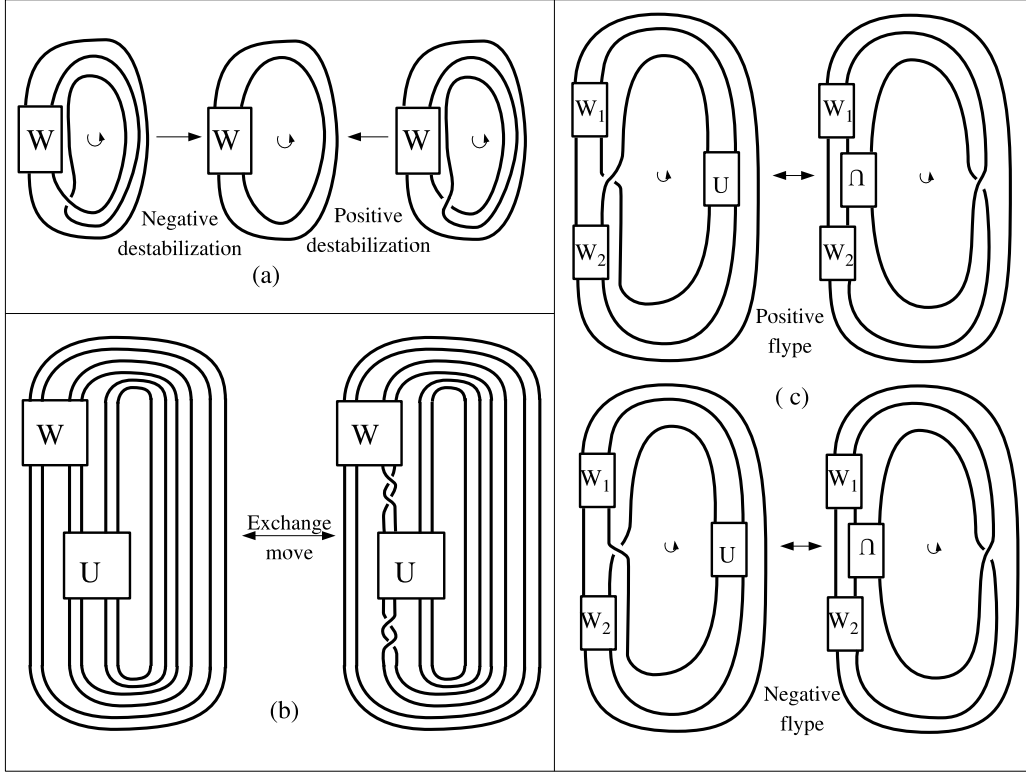


Figure 4: Destabilization, an exchange move and braid preserving flypes are depicted respectively in illustrations (a), (b) and (c). In particular, in (b) we have illustrated an exchange move on a 6-braid so as to emphasize the occurrence of the full twists $\tau_{[3,4]}$ & $\tau_{[3,4]}^{-1}$. The block labels correspond to the associated braid syllable. In particular, in (c) when $U = \sigma_{n-1}^p$ we have an elementary flype. The labels in the braid boxes correspond to the associated braid syllable.

“2-strand crossing”.)

X admits an exchange move if $\beta(X)$ is of the form WU where W (respectively U) is a word of \mathcal{W}^t (respectively \mathcal{U}^s) for some integers $s \leq t$. The n -braid Y that is associated with the cyclic word $W\tau_{[s,t+1]}U\tau_{[s,t+1]}^{-1}$, where $\tau_{[s,t+1]}$ is a full (positive or negative) twist on the s through $(t+1)$ strands, is *exchange related* to X . (See Figure 4(b).)

We further refine our concept of exchange move. Suppose X admits an exchange move. Then $\beta(X)$ is of the form WU where W is in \mathcal{W}^t and U is in \mathcal{U}^s . The reader should observe that if W is a word in \mathcal{W}^t then it is also a word in $\mathcal{W}^{t'}$ for any $t < t' \leq n-2$. Similarly, U is also a word in $\mathcal{U}^{s'}$ for $2 \leq s' < s$. Since the specification of t and s determine which strands are to be used in the full twist $\tau_{[s,t+1]}$ it is useful to have a canonical way of viewing the words W and U . Thus, we say the product $WU = \beta(X)$ corresponds to a *thin exchange move* if for all choices of s and t with $s \leq t$ we have $t-s$ minimal. In Figure 4(b) we have $s = 3 = t$, so

this illustration corresponds to a thin exchange move. It is clear that if X admits an exchange move then it has a thin exchange move.

Continuing, $\beta(X)$ admits a (braid preserving) flype if it is of the form $W_1 U W_2 \sigma_{n-1}^{\pm 1}$ where $W_1, W_2 \in \mathcal{W}^{n-2}$ and $U \in \mathcal{U}^s$ for some integers $s \leq n-1$. When $s = n-1$ we have $\beta(X)$ of the form $W_1 \sigma_{n-1}^p W_2 \sigma_{n-1}^{\pm 1}$ where $p \in \mathbb{Z} - \{0\}$ and $\beta(X)$ admits an elementary (braid preserving) flype. The n -braid Y which has $\beta(Y) = W_1 \sigma_{n-1}^{\pm 1} W_2 \sigma_{n-1}^p$ is elementary flype related to X .

We say that $\mathcal{B}_n(X)$ admits a destabilization, exchange move, flype, or elementary flype, respectively, if there exists a braid representative $X' \in \mathcal{B}_n(X)$ which admits a destabilization, exchange move, flype, or elementary flype, respectively. (As mentioned, if $\mathcal{B}_n(X)$ admits an exchange move then there will be a $X'' \in \mathcal{B}_n(X)$ that will admit a thin exchange move.)

Remark 1 The reader should notice that when $s = t + 1$ then the product WU represents a composite link. When $s \geq t + 2$ then the product WU is a split link. We will assume that $\beta(X)$ is neither composite nor split. The reader should further observe that when $\beta(X)$ is of the form WU with $W \in \mathcal{W}^t$, $U \in \mathcal{U}^s$ and $t - s < n - 3$ then $\beta(X)$ admits infinitely many distinct exchange moves. To take an illustrative construction, suppose we have a n -braid X with $\beta(X) = WU$ where with $W \in \mathcal{W}^t$ and $U \in \mathcal{U}^s$ and $s > 2$. Let α be any braid word in \mathcal{W}^{s-1} . Then for the cyclic word WU we have

$$WU = W\alpha\alpha^{-1}U = W\alpha U\alpha^{-1} = [\alpha^{-1}W\alpha]U = W'U'$$

where still $W' \in \mathcal{W}^t$ but $U' \in \mathcal{U}^s$. Now viewing $W' \in \mathcal{W}^t$ and $U' \in \mathcal{U}^{s-1}$ we perform an exchange move on $W'U'$ to obtain $W'\tau_{[s-1,t+1]}U'\tau_{[s-1,t+1]}^{-1}$. For a judicious choose of W , U and α we can insure that $W'\tau U'\tau^{-1}$ is not braid isotopic to $W\tau U\tau^{-1}$. Thus, by varying α —for example, taking powers of a fixed α —we can produce infinitely many exchange moves. From this example it becomes clear that the exchange moves that will always “respect braid isotopy” are the thin exchange moves. That is, when performing a thin exchange move we are requiring that the full twist τ be on the least number of strands. In our results this will have the implication that we will be able to recognize when two braids X and Y are related by an thin exchange move. Finally, we observe that this phenomenon is not an issue for elementary flypes. As with exchange moves, it is true that we have a similar sequence of cyclic equalities

$$W_1 \sigma_{n-1}^p W_2 \sigma_{n-1}^{\pm 1} = W_1 \alpha \alpha^{-1} \sigma_{n-1}^p W_2 \sigma_{n-1}^{\pm 1} = [W_1 \alpha] \sigma_{n-1}^p [\alpha^{-1} W_2] \sigma_{n-1}^{\pm 1} = W'_1 \sigma_{n-1}^p W'_2 \sigma_{n-1}^{\pm 1}$$

for any word α using generators in \mathcal{W}^{n-3} . But, after the elementary flype we also have

$$W'_1 \sigma_{n-1}^{\pm 1} W'_2 \sigma_{n-1}^p = [W_1 \alpha] \sigma_{n-1}^{\pm 1} [\alpha^{-1} W_2] \sigma_{n-1}^p = W_1 \alpha \alpha^{-1} \sigma_{n-1}^{\pm 1} W_2 \sigma_{n-1}^p = W_1 \sigma_{n-1}^{\pm 1} W_2 \sigma_{n-1}^p.$$

Thus, although there are infinitely many words admitting the same elementary flype, up to braid isotopy all of these elementary flypes relate to the same two closed n -braids. \diamond

Remark 2 We also observe that having $\beta(X) = WU$ with $W \in \mathcal{W}^t$, $U \in \mathcal{U}^s$ and $s > 2$ is not a unique format illustrating even a thin exchange move. Specifically, suppose $W = W'\alpha$ with $W' \in \mathcal{W}^t$ and $\alpha \in \mathcal{W}^{n-s-1}$. Then α and U commute in the braid group, and the closed braids having cyclic words $[W\alpha]U$ and $[\alpha W]U$ are braid isotopic. Moreover, the closed braids having cyclic words $[W\alpha]\tau_{[s,t+1]}^{-1}U\tau_{[s,t+1]}^{-1}$ and $[\alpha W]\tau_{[s,t+1]}^{-1}U\tau_{[s,t+1]}^{-1}$ are braid isotopic. Thus, it is convenient to equate $[W\alpha]U$ and $[\alpha W]U$ as admitting the same exchange move. \diamond

Remark 3 We can apply the discussion in the previous remarks to an n -braid $\beta(X) = W\mathbf{X}^{\epsilon \cdot 2}$ where $\mathbf{X}^{\epsilon \cdot 2}$ is a 2-strand crossing so as to have X admitting a double destabilization. Notice that $X = W\mathbf{X}^{\epsilon \cdot 2}$ also admits an exchange move and is exchange equivalent to $X' = W\tau_{[(n-2),(n-1)]}^p\mathbf{X}^{\epsilon \cdot 2}\tau_{[(n-2),(n-1)]}^{-p}$ for $p \in \mathbb{Z}$. Now the reader should notice two features. First, since $W\mathbf{X}^{\epsilon \cdot 2}$ has 2 parallel strands we can slide the $\tau_{[(n-2),(n-1)]}^p$ -twist through these strands to cancel the $\tau_{[(n-2),(n-1)]}^{-p}$ -twists. Thus, X and X' are braid isotopic. Second, by destabilizing a single strand of the 2-strands of $\mathbf{X}^{\epsilon \cdot 2}$ we have a means by which to produce infinitely distinct conjugacy classes. This phenomena of X destabilizing to infinitely many possible distinct conjugacy classes was investigated by A. V. Malyutin [M1, M2]. (The author wishes to thank the referee for alerting him to this body of work.) \diamond

A long standing problem (Problem 1.84 in [K]) is determining when an n -braid equivalence class $\mathcal{B}_n(X)$ contains a braid X that admits either a destabilization, an exchange move, or an elementary flype. Our main result (Theorem 4) states that there is a simple algorithmic method for making these determinations. To understand this algorithm we need to consider our representation of X in \mathbf{H} anew using ‘rectangular diagrams’.

Notational conventions— Our discussion will also require extensive use of arcs joined together at common endpoints to form edge-paths that are homeomorphic to either a closed interval or circle. We thus introduce notation for *ordered union* and *cyclic ordered union*. So, for arcs $\{a_1, \dots, a_l\}$ with a_i sharing a single endpoint with a_{i+1} , $1 \leq i < l$, we have the ordered union $a_1 \dot{\cup} a_2 \dot{\cup} \dots \dot{\cup} a_l$ being the edge-path homeomorphic to a closed interval that is obtained by adjoining a_i to a_{i+1} at their common endpoint, $1 \leq i < l$. If a_1 and a_l also share a common endpoint then the cyclic ordered union $a_1 \dot{\cup} \dots \dot{\cup} a_l \dot{\cup}$ is homeomorphic to a circle.

Finally, in our use of angle parameters θ we have identify S^1 with the quotient space $[0, 2\pi]/0 \sim 2\pi$. Since S^1 is given the orientation that corresponds to winding positively around the z – axis, the points on S^1 are cyclically ordered and it makes sense to talk about the oriented angle interval $[\theta_1, \theta_2] \subset S^1$ that starts at θ_1 and ends at θ_2 . The arc length $||[\theta_1, \theta_2]||$ will necessarily be less than 2π .

1.2 Rectangular diagrams and main results.

A *horizontal arc*, $h \subset C_1$, is any arc having parametrization $\{(1, t, z_0) \mid t \in [\theta_1, \theta_2]\}$. The *horizontal position* of h is the fixed constant z_0 . The *angular support* of h is the angle interval $[\theta_1, \theta_2] \subset S^1$. Horizontal arcs inherit a natural orientation from the forward direction of the θ coordinate. A *vertical arc*, $v \subset H_{\theta_0}$, is any arc having parametrization $\{(r(t), \theta_0, z(t)) \mid 0 \leq t \leq 1, r(0) = r(1) = 1; \text{ and } r(t) > 1, \frac{dz}{dt} > 0 \text{ for } t \in (0, 1)\}$, where $r(t)$ and $z(t)$ are real-valued functions that are continuous on $[0, 1]$ and differentiable on $(0, 1)$. The *angular position* of v is θ_0 . The *vertical support* of v is the interval $[z(0), z(1)]$. (We remark that the parametrization of the the vertical arcs will not be used in assigning orientation to the vertical arcs.)

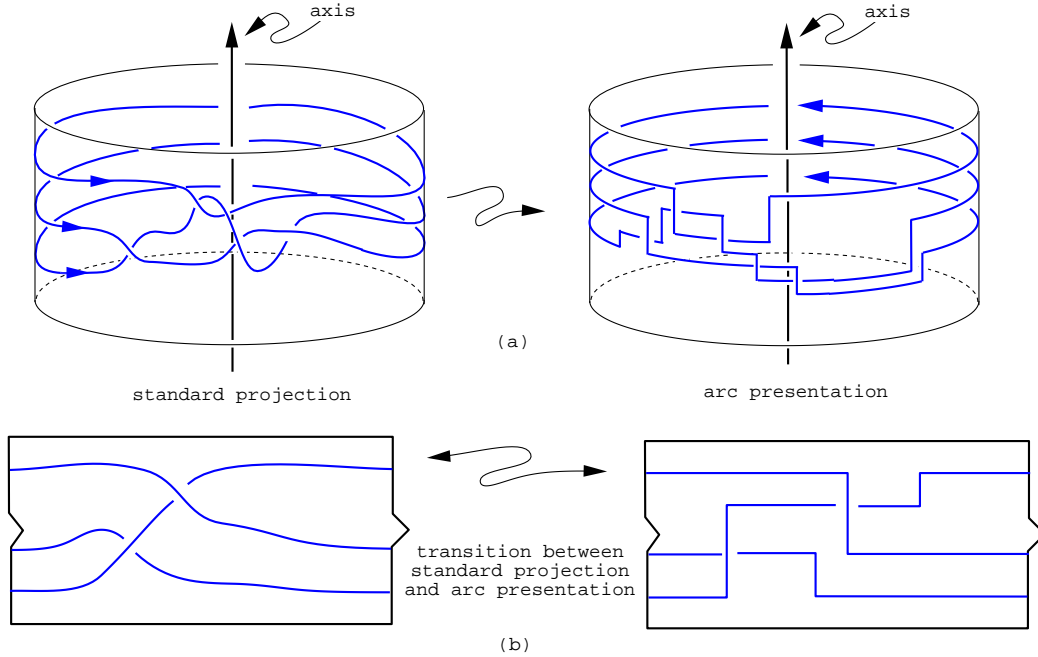


Figure 5:

Let \mathcal{X} be an oriented link type in S^3 . $X^\eta \in \mathcal{X}$ is an *arc presentation* if each component Y of X^η is a cyclic union of arcs $h_1^Y \cup v_1^Y \cup \dots \cup h_k^Y \cup v_k^Y \cup$ with k necessarily varying in value between components and:

1. each h_i^Y , $1 \leq i \leq k$; is an oriented *horizontal arc* having orientation agreeing with the Y ,
2. each v_i^Y , $1 \leq i \leq k$, is a *vertical arc* having orientation agreeing with Y ;
3. $h_i^Y \cap v_j^Y \subset \partial h_i^Y \cap \partial v_j^Y$, and for $1 \leq i \leq k$ this intersection is a single point when $j \pmod k = \{i, i-1\}$, otherwise it is empty;
4. the horizontal position of each horizontal arc is distinct over all components of X^η ;

5. the angular position of each vertical arc is distinct over all components of X^η .

For a given arc presentation X^η there is a cyclic order to the horizontal positions of the h_i 's, as determined by their occurrence on the \mathbf{A} , and a cyclic order to the angular position of the v_j 's, as determined by their occurrence in \mathbf{H} . It is clear that given two arc presentations with identical cyclic order for horizontal positions and angular positions there is an ambient isotopy between the two presentations that corresponds to re-scaling of the horizontal positions and angular positions along with the vertical and angular support of the arcs in the presentations. Thus, we will think of two arc presentations as being equivalent if the cyclic orderings of their horizontal positions and vertical positions are equivalent.

We define the *complexity* of the arc presentation X^η , $\mathcal{C}(X^\eta)$, as being the number of vertical arcs.

Given a closed n -braid X and a corresponding standard projection $\pi(X)$ we can easily produce a (not necessarily unique) arc presentation X^η as illustrated by the transition in Figure 5. Clearly, there is also the transition from an arc presentation X^η to an n -braid X with a standard projection $\pi(X)$. We will use the notation $X \xrightarrow{N} X^\eta$ or $X^\eta \xrightarrow{B} X$ to indicate these two presentation transitions.

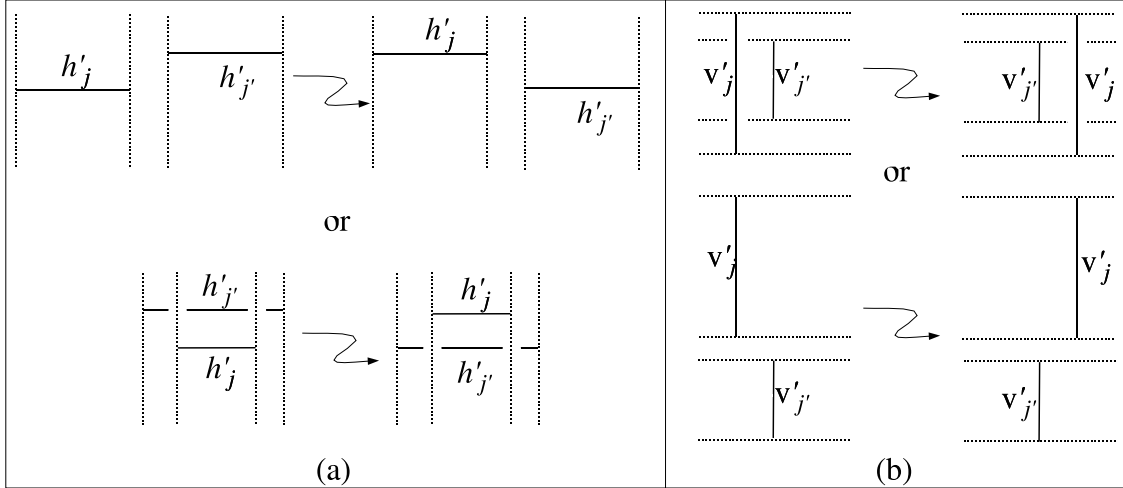


Figure 6: Illustration (a) corresponds to horizontal exchange moves and (b) corresponds to vertical exchange moves. In each of the illustrated sequences dotted arcs are used to indicate that the adjoining vertical arcs (for (a)) and horizontal arcs (for (b)) have two possible ways of attaching themselves to the labeled solid arc.

We next define *elementary moves* on an arc presentation X^η . To setup these moves we let

$$X^\eta = \bigsqcup_{Y \in X} (h_1^Y \dot{\cup} v_1^Y \dot{\cup} \dots \dot{\cup} h_k^Y \dot{\cup} v_k^Y \dot{\cup})$$

with: z_i^Y the horizontal position for h_i^Y ; $[\theta_{i_1}^Y, \theta_{i_2}^Y] \subset S^1$ the angular support for h_i^Y ; θ_i^Y the angular position for v_i^Y ; and $[z_{i_1}^Y, z_{i_2}^Y]$ the vertical support for v_i^Y (where $1 \leq i \leq k^Y$ in all statements). It is convenient at times to drop the uses of the Y superscript. The reader should also be alert to the use of the subscript index variable to indicate needed associations, e.g. h_i has angular support $[\theta_{i_1}, \theta_{i_2}]$ whereas h_j has angular support $[\theta_{j_1}, \theta_{j_2}]$.)

Horizontal exchange move—The move comes in two flavors. The first allows us to take the horizontal arc of X^η that is of maximal (respectively, minimal) horizontal position and, without altering its angular support, reposition it to be of minimal (respectively, maximal) horizontal position. The attached vertical arcs are adjusted in a corresponding manner. The second flavor considers two distinct horizontal arcs, h_i and h_j , of X^η that are, first, *consecutive* and, second, *nested*. Namely, first, their associated horizontal positions, z_i & z_j , are consecutive in the ordering of the horizontal positions. And second, either $[\theta_{i_1}, \theta_{i_2}] \subset [\theta_{j_1}, \theta_{j_2}]$, or $[\theta_{j_1}, \theta_{j_2}] \subset [\theta_{i_1}, \theta_{i_2}]$, or $[\theta_{i_1}, \theta_{i_2}] \cap [\theta_{j_1}, \theta_{j_2}] = \emptyset$, i.e. nested. Then we can locally alter X^η by replacing the edgepaths $v_{i-1} \dot{\cup} h_i \dot{\cup} v_i$ and $v_{j-1} \dot{\cup} h_j \dot{\cup} v_j$ with, respectively, $v'_{i-1} \dot{\cup} h'_i \dot{\cup} v'_i$ and $v'_{j-1} \dot{\cup} h'_j \dot{\cup} v'_j$, where for the corresponding horizontal positions we have $z'_i = z_j$ and $z'_j = z_i$, and the vertical support for v'_{i-1} , v'_i , v'_{j-1} and v'_j are adjusted in a corresponding manner. The boundary endpoints of these edgepaths are fixed under this alteration and all other vertical and horizontal arcs of X^η are fixed. (See Figure 6(a).)

Vertical exchange move—Let v_i and v_j be two distinct vertical arcs of X^η that are, again, *consecutive* and *nested*. Namely, the angular positions θ_i and θ_j are consecutive in the cyclic ordering of the vertical arcs of X^η . And, either $[z_{i_1}, z_{i_2}] \subset [z_{j_1}, z_{j_2}]$, or $[z_{j_1}, z_{j_2}] \subset [z_{i_1}, z_{i_2}]$, or $[z_{i_1}, z_{i_2}] \cap [z_{j_1}, z_{j_2}] = \emptyset$, i.e. nested. Then we can locally alter X^η by replacing the edgepaths $h_{i-1} \dot{\cup} v_i \dot{\cup} h_i$ & $h_{j-1} \dot{\cup} v_j \dot{\cup} h_j$ with, respectively, $h'_{i-1} \dot{\cup} v'_i \dot{\cup} h'_i$ & $h'_{j-1} \dot{\cup} v'_j \dot{\cup} h'_j$ where for the corresponding angular positions we have $\theta'_i = \theta_j$ and $\theta'_j = \theta_i$, and the angular support for h'_{i-1} , h'_i , h'_{j-1} and h'_j are adjusted in a corresponding manner. Again, the boundary endpoints of these edgepaths are fixed under this alteration and all other vertical and horizontal arcs of X^η are fixed. (See Figure 6(b).)

Horizontal simplification—Let h_i and h_{i+1} be two horizontal arcs that are consecutive (as previously defined in the horizontal exchange move), and are adjacent to a common vertical arc v_i so that $h_i \dot{\cup} v_i \dot{\cup} h_{i+1} \dot{\cup} v_{i+1}$ is an edgepath on a component of X^η . Then we can locally alter X^η by replacing $h_i \dot{\cup} v_i \dot{\cup} h_{i+1} \dot{\cup} v_{i+1}$ with an edgepath $h'_i \dot{\cup} v'_{i+1}$ where: the horizontal position of h'_i is z_i ; the angular support of h'_i is $[\theta_{i_1}, \theta_{i_2}] \cup [\theta_{(i+1)_1}, \theta_{(i+1)_2}]$; and, the angular position of v'_{i+1} is θ_{i+1} . (Notice that $\theta_{i_2} = \theta_{(i+1)_1}$.) Since our vertical support notation does not have a correspondence to the orientation of vertical arcs, the vertical support of v'_{i+1} must be specified with some care: it is in fact the closure on \mathbf{A} of the interval $\{(z_{i_1}, z_{i_2}) \cup (z_{(i+1)_1}, z_{(i+1)_2})\} \setminus \{(z_{i_1}, z_{i_2}) \cap (z_{(i+1)_1}, z_{(i+1)_2})\}$. Again, the boundary endpoints of these edgepaths are fixed under this alteration and all other vertical and horizontal arcs of X^η are fixed.

Vertical simplification—Let v_i and v_{i+1} be two vertical arcs that are consecutive (as previously defined in the vertical exchange move), and are adjacent to a common horizontal arc h_i so that $v_i \dot{\cup} h_i \dot{\cup} v_{i+1} \dot{\cup} h_{i+1}$ is an edgepath on a component of X^η . Then we can locally alter X^η by replacing $v_i \dot{\cup} h_i \dot{\cup} v_{i+1} \dot{\cup} h_{i+1}$ with an edgepath $v'_i \dot{\cup} h'_{i+1}$ where: the angular position of v'_i is θ_i ; the vertical support of v'_i is the closure on \mathbf{A} of the interval $\{(z_{i_1}, z_{i_2}) \cup (z_{(i+1)_1}, z_{(i+1)_2})\} \setminus \{(z_{i_1}, z_{i_2}) \cap (z_{(i+1)_1}, z_{(i+1)_2})\}$; the horizontal position of h'_{i+1} is z_{i+1} ; and the angular support of h'_{i+1} is $[\theta_{i_1}, \theta_{i_2}] \cup [\theta_{(i+1)_1}, \theta_{(i+1)_2}]$. As before, the boundary endpoints of these edgepaths are fixed under this alteration and all other vertical and horizontal arcs of X^η are fixed.

Given an arc presentation X^η we notice that for any sequence of elementary moves applied to X^η , the complexity measure $\mathcal{C}(X^\eta)$ is non-increasing. That is, any sequence of elementary moves which includes the uses of either horizontal or vertical simplification will be *monotonically simplified*.

One would hope that for a closed n -braid X which admits, respectively, a destabilization, exchange move, or elementary flype, there exists a sequence of elementary moves to the arc presentation X^η (coming from $X \xrightarrow{N} X^\eta$) such that for the resulting arc presentation X'^η , the closed n -braid X' coming from $X'^\eta \xrightarrow{B} X'$ admits, respectively, a destabilization, exchange move or elementary flype (as seen from the standard projection $\pi(X')$). Unfortunately, this is too good to be true. In order to produce an X' that admits the assumed isotopy it may be necessary to increase the number of arcs in the arc presentation. At first glance this seems to disturb our ability to monotonically simplify. However, it is possible to control the manner in which we introduce additional arcs in the arc presentation to monotonically simplify using an altered complexity measure.

To accomplish this controlled addition of arcs we introduce the notion of "shearing intervals". For a given arc presentation X^η let $\mathcal{I} \subset S^1$ be a union of disjoint closed angle intervals of the form $[\vartheta^k - \epsilon^k, \vartheta^k + \epsilon^k]$, $0 \leq k \leq l$ such that for $\vartheta \in \mathcal{I}$ we have that $H_\vartheta \in \mathbf{H}$ contains no vertical arc of X^η . The value of l will be 1, 2, or 3 when discussing, respectively, destabilization, exchange move, or elementary flype. Then $X^\eta_{\mathcal{I}}$ is an arc presentation X^η along with a specification of where the *shearing intervals* of \mathcal{I} are to be initially positions in S^1 . We require that this initial positioning of \mathcal{I} be away from the vertical arcs of X^η . As such notice that the positioning of a component of \mathcal{I} is characterized by which vertical arcs it lies between. Thus, once we specify how many components \mathcal{I} should contain, up to angular rescaling there are only finitely many possible initial $X^\eta_{\mathcal{I}}$ for a given X^η .

We define the complexity measure $\mathcal{C}(X^\eta_{\mathcal{I}})$ to be the number of vertical arcs in the angle interval(s) $S^1 \setminus \mathcal{I}$. The reader should notice that for an initial $X^\eta_{\mathcal{I}}$ we have $\mathcal{C}(X^\eta_{\mathcal{I}}) = \mathcal{C}(X^\eta)$.

We notice that our previous elementary moves—horizontal or vertical exchange moves and simplification—can also be applied to $X^\eta_{\mathcal{I}}$ in the angular intervals of $S^1 \setminus \mathcal{I}$ so as to monotonically

simplify $X_{\mathcal{I}}^\eta$ with respect to its complexity. Specifically, if any of the edgepaths used in the description of our elementary moves are totally contained in a component of $S^1 \setminus \mathcal{I}$ then applying the move to X^η can be seen as applying to move to $X_{\mathcal{I}}^\eta$. So for horizontal or vertical exchange moves $\mathcal{C}(X_{\mathcal{I}}^\eta)$ is unchanged. Similarly, for the simplification moves the complexity measure of the resulting $X_{\mathcal{I}}^\eta$ decreases.

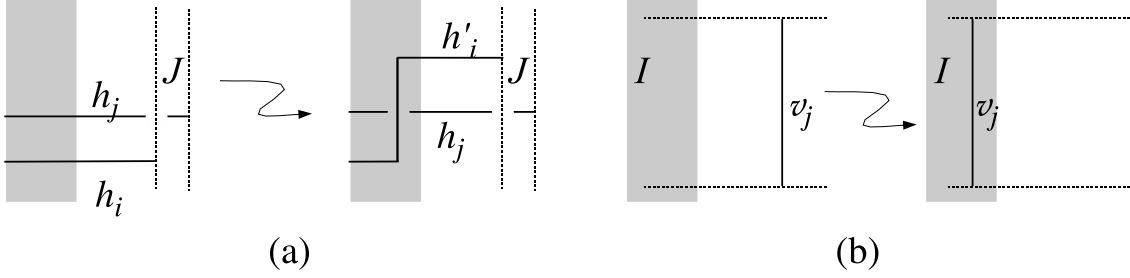


Figure 7: Figure (a) illustrates a shear-horizontal exchange and (b) illustrates a shear-vertical simplification. Again, in each of the illustrated sequences dotted arcs are used to indicate that the adjoining vertical arcs (for (a)) and horizontal arcs (for (b)) have two possible ways of attaching themselves to the labeled solid arc.

We now add two new elementary moves that utilizes the intervals of \mathcal{I} .

Shear horizontal exchange move—We refer to Figure 7(a). Let h_i and h_j be two horizontal arcs of $X_{\mathcal{I}}^\eta$ that are *consecutive* and *nested* with respect to \mathcal{I} . That is, first, for some angle interval $J \subset S^1 \setminus \mathcal{I}$ we have that: a) h_i and h_j intersect J ; and, b) over all of the horizontal arcs of $X_{\mathcal{I}}^\eta$ which intersect J , h_i and h_j have consecutive horizontal positions in the ordering along \mathbf{A} . Second, the angular support of $h_i \cap J$ is contained inside the angular support of $h_j \cap J$. Then we can interchange the horizontal position of these two arcs. This is achieved by a *horizontal shear* inside an interval of \mathcal{I} —the introduction of a vertical (contained in \mathcal{I}) and a horizontal arc as shown in Figure 7(a)—to the portion of h_i and h_j that is contained in \mathcal{I} . Thus, when we consider the resulting arc presentations an original horizontal exchange move is realizable. We abuse notation still referring to the resulted arc presentation as $X_{\mathcal{I}}^\eta$. Notice that $\mathcal{C}(X_{\mathcal{I}}^\eta)$ remains constant.

Shear vertical simplification—We refer to Figure 7(b). Let v_j be a vertical arc of $X_{\mathcal{I}}^\eta$ that is *consecutive* with respect to \mathcal{I} . That is, for an interval $I \in \mathcal{I}$ there is no vertical arc whose angular position is between v_j and I . We can then push v_j into I . Again, we abuse notation by referring to the resulted arc presentation as $X_{\mathcal{I}}^\eta$. Notice that $\mathcal{C}(X_{\mathcal{I}}^\eta)$ is decreased by a count of one.

From now on we refer to horizontal or vertical exchange moves and simplification along with shear horizontal exchange moves and shear vertical simplification as our collection of elementary moves on arc presentations with shearing intervals. Since our notation, $X_{\mathcal{I}}^\eta$, is for

an arc presentation along with unchanging shearing intervals under elementary moves, it is very convenient to abuse notation and use $X_{\mathcal{I}}^{\eta}$ when making set operation statements involving the underlying arc presentation. For example, by $X_{\mathcal{I}}^{\eta} \cap C_1$ we will mean $X^{\eta} \cap C_1$.

To connect the dots, for a given X^{η} and fixing the number of angle intervals in \mathcal{I} , up to re-scaling, there are only finitely many initial $X_{\mathcal{I}}^{\eta}$, i.e. only finitely many *combinatorial distinct* initial $X_{\mathcal{I}}^{\eta}$. The complexity measure $\mathcal{C}(X_{\mathcal{I}}^{\eta})$ is just a count on the number of vertical arcs in $S^1 \setminus \mathcal{I}$ and that elementary moves on a $X_{\mathcal{I}}^{\eta}$ never increases the number of vertical arcs in $S^1 \setminus \mathcal{I}$. So starting with $\mathcal{C}(X^{\eta})$ vertical arcs in the intervals of $S^1 \setminus \mathcal{I}$, after any sequence of elementary moves there is a finite number of possible combinatorial distinct rectangular (non-closed) braid presentations in the intervals of $S^1 \setminus \mathcal{I}$. (Again, “combinatorial distinct” refers to equivalency up to rescaling of angle and height positions of arcs.) Thus, starting with an initial $X_{\mathcal{I}}^{\eta}$ (having no vertical arcs in \mathcal{I}) there are only finitely many possible rectangular diagrams occurring in the intervals $S^1 \setminus \mathcal{I}$ after any sequence of elementary moves to $X_{\mathcal{I}}^{\eta}$ up to our combinatorial equivalence.

Due to the finite number of rectangular diagrams occurring in the $S^1 \setminus \mathcal{I}$ intervals, for any sequence of elementary moves on $X_{\mathcal{I}}^{\eta}$ that produces only combinatorial distinct diagrams in $S^1 \setminus \mathcal{I}$, a bounded number of additional horizontal and vertical arcs inside the intervals of \mathcal{I} through the application of shear horizontal exchange moves and shear vertical simplifications will be introduced. However, it is possible to have arbitrarily long sequences of elementary moves on $X_{\mathcal{I}}^{\eta}$ containing the occurrence of same combinatorial distinct rectangular diagrams in the intervals $S^1 \setminus \mathcal{I}$ arbitrarily many times. For example, starting with a fixed $X_{\mathcal{I}}^{\eta}$ one could produce a finite cyclic sequence of rectangular diagrams—starting and ending at the same fixed $X_{\mathcal{I}}^{\eta}$ —all having the same complexity measure by applying a sequence of horizontal, vertical and shear horizontal exchange moves. If we repeat such a sequence any number of times we can create the canceling α -braiding phenomena of Remarks 1 & 2, or the canceling τ -braiding phenomena of Remark 3. We concluded that although what can occur in the \mathcal{I} intervals may be infinite, what can occur in $S^1 \setminus \mathcal{I}$ is finite, and recognizing when a closed braid admits one of our isotopies will depend on interrupting the diagrams in $S^1 \setminus \mathcal{I}$.

We are now in a position to state our main results.

Theorem 4 *Let X be a closed n -braid such that $\mathcal{B}_n(X)$ admits, respectively, a destabilization, exchange move or elementary flype. Consider any arc presentation coming from the a presentation transition $X \xrightarrow{N} X^{\eta}$. Then there exists a set of intervals \mathcal{I} and a sequence of arc presentations*

$$X_{\mathcal{I}}^{\eta} = X^0 \rightarrow X^1 \rightarrow \dots \rightarrow X^l = X'_{\mathcal{I}}{}^{\eta}$$

such that:

1. If $\mathcal{B}_n(X)$ admits, respectively, a destabilization, exchange move, flype or elementary flype then \mathcal{I} has, respectively, one, two or three intervals.
2. X^{i+1} is obtained from X^i via one of the elementary moves. All of these moves are with respect to the intervals of \mathcal{I} .
3. The closed n -braid obtained from the presentation transition $X_{\mathcal{I}}^{\eta} \xrightarrow{B} X'$ admits, respectively, a destabilization, exchange move, flype or elementary flype (as seen from the standard projection $\pi(X')$).

We observe that $\mathcal{C}(X^{i+1}) \leq \mathcal{C}(X^i)$ for $0 \leq i \leq l$ for all applications of our elementary moves. In particular, if $X^i \rightarrow X^{i+1}$ corresponds to a horizontal, vertical or shear horizontal exchange move then $\mathcal{C}(X^i) = \mathcal{C}(X^{i+1})$. If it corresponds to a horizontal, vertical or shear vertical simplification then $\mathcal{C}(X^{i+1}) < \mathcal{C}(X^i)$, i.e. monotonically simplified.

As previously remarked, we will restrict ourselves to only three possible choices for \mathcal{I} —it has one, two or three intervals—and we recall the positioning of these intervals is characterized by which vertical arcs of a initially given X^{η} they lie between. Thus, there are only finitely many possible initial $X_{\mathcal{I}}^{\eta}$. Also, our previous remarks gives us that there are only finitely many resulting $X_{\mathcal{I}}^{\eta}$ after elementary moves. The production of such a finite set is easily seen as algorithmic. Therefore, Theorem 4 implies the following corollary.

Corollary 5 *There exists an algorithm for deciding whether a closed n -braid is braid isotopic to one that admits either a destabilization, exchange move, or elementary flype.*

The construction of our algorithmic solutions comes from utilizing the braid foliation machinery that was first developed in [BF, BM1, BM2, BM3] and further refined in the beautiful work of I.A. Dynnikov [D].

In [M1], exploiting the interpretation of n -braids as elements of the mapping class group of the n -punctured disc and Nielsen-Thurston's theory, an alternate algorithm is established for determining when a closed braid admits a destabilization. This algorithm is based upon an analysis of the action of the mapping class group on the geodesics of the n -punctured disc endowed with a fixed hyperbolic metric.

In a strict sense, Theorem 4 is an existence result—it tells us if a given braid admits a given move. Based upon Remarks 1 & 3 we know that determining whether two fixed braids are related by a particular move is problematic. But, by paying close attention to the machinery in the proof of Theorem 4 we can make such a determination in the cases of a double destabilization, thin exchange move, and elementary flype. Thus, we have the following theorem.

Theorem 6 *There exists an algorithm for deciding whether closed braid Y is related to X by a thin exchange move, an elementary flype. And, there exists an algorithm for deciding whether X admits a double destabilization.*

ACKNOWLEDGMENTS—The author wishes to thank the referee’s considerable input and energy. Specifically, the author was alerted to the connections to the work of A.V. Malyutin by the referee; and, Theorem 6 is essentially the referee’s formulation. The author also acknowledges NSF partial support through grant #DMS 0306062.

2 The cylinder machinery.

2.1 Destabilizing, exchange and flyping discs.

Our first objective is to give a geometric characterization for recognizing when a closed n -braid is braid isotopic to one that admits either a destabilization, exchange move or elementary flype. All geometric characterizations will depend on the existence of a specified embedded disc. Our characterizations will, in fact, occur in pairs: one for the braid presentation and one for the arc presentation.

All of our geometric characterizing discs, Δ_ϵ will be *above the braid*. That is, $\Delta_\epsilon = D_{+1} \cup N$ where:

1. D_{+1} is the disc in a plane $z = z_{max}$ having $0 \leq r \leq 1, \theta \in S^1$ and z_{max} being a constant greater the horizontal positions of all the horizontal arcs of X^η . (Thus, $D_{+1} \cap X^\eta = \emptyset$.)
2. Δ_ϵ is oriented so that D_{+1} necessarily intersects \mathbf{A} geometrically and algebraically $+1$ at a *vertex* point \mathbf{v}_{max} .
3. N is an annulus having $r \geq 1$ for all its points.

Destabilizing disc—(Braid presentation) Let X be a closed n -braid which admits a destabilization, i.e. the corresponding braid word $\beta(X) = W\sigma_{n-1}^{\pm 1}$ with $W \in \mathcal{W}^{n-2}$. Then there exists a *destabilizing disc* Δ_d having the following properties.

- D-a. $\partial\Delta_d = \alpha_h \dot{\cup} \alpha_v^\partial \dot{\cup}$ where we have the *horizontal boundary* $\alpha_h \subset X$ and the *∂ -vertical arc* $\alpha_v^\partial \subset H_{\theta\partial}$ for some $H_{\theta\partial} \in \mathbf{H}$
- D-b. $\Delta_d \cap X = \alpha_h$.
- D-c. Δ_d transversely intersects \mathbf{A} at a single vertex point \mathbf{v}_{max} .

D-d. $a_\theta = \Delta_d \cap H_\theta$ is a single arc having an endpoint on α_h and \mathbf{v}_{max} as the other endpoint, for $\theta \in S^1 - \{\theta^\partial\}$. If $\theta = \theta^\partial$ then $\alpha_v^\partial \subset a_{\theta^\partial} (= \Delta_d \cap H_{\theta^\partial})$. We require all leaves to be homeomorphic to the unit interval of \mathbb{R} . To summarize, the braid fibration induces a *radial foliation* on Δ_d .

If X' is braid isotopic to X then we can extend the braid isotopy which takes X to X' to an ambient isotopy of $S^3 \setminus \mathbf{A}$. This ambient isotopy takes Δ_d to a destabilizing disc for X' , i.e. properties D-a through D-d are still satisfied. Thus, every n -braid representative of $\mathcal{B}_n(X)$ will have a destabilizing disc.

Further analysis of the leaves containing the ∂ -vertical arcs is useful. Specifically, let a_{θ^∂} be a leaf in the radial foliation of Δ_d that contains an ∂ -vertical arc α_v^∂ . If there is an angle interval $[\theta', \theta^\partial] (\subset S^1)$ such that for any $\theta \in [\theta', \theta^\partial]$ pushing the leaf a_θ forward in the radial foliation to a_{θ^∂} corresponds to a homeomorphism between a_θ and a_{θ^∂} then we say that α_v^∂ has a *front edge*. Similarly, if there is an angle interval $[\theta^\partial, \theta'] (\subset S^1)$ such that for any $\theta \in [\theta^\partial, \theta']$ pushing the leaf a_θ backwards in the radial foliation to a_{θ^∂} corresponds to a homeomorphism between a_θ and a_{θ^∂} then we say that α_v^∂ has a *back edge*. Notice that by the definition of Δ_d , α_v^∂ has either a front edge or a back edge, but not both.

Without loss of generality we make the convenient assumption that if Δ_d is a positive (respectively, negative) destabilizing disc then the ∂ -vertical arc is a front (respectively, back) edge. Notice that an edge assignment is stable under braid isotopy.

(Arc presentation) Again with $\beta(X) = W\sigma_{n-1}^{\pm 1}$, we consider the arc presentation coming from the transition $X \xrightarrow{N} X^\eta$. Then we also have a disc, which is call an *obvious destabilizing disc*, that has the following properties.

DA-a. $\partial\Delta_d = h_1 \dot{\cup} v_1 \dot{\cup} h_2 \dot{\cup} v_2^\partial \dot{\cup}$ where v_1 is a vertical arc of X^η , h_1 & h_2 are contained in horizontal arcs of X^η , and $v_2^\partial \subset H_{\theta^\partial}$ for $H_{\theta^\partial} \in \mathbf{H}$ is the ∂ -vertical arc of Δ_d .

DA-b. $\Delta_d \cap X^\eta = h_1 \dot{\cup} v_1 \dot{\cup} h_2$.

DA-c. Δ_d transversely intersects \mathbf{A} at a single vertex point \mathbf{v}_{max} , the *vertex* of the foliation on Δ_d .

DA-d. $a_\theta = \Delta_d \cap H_\theta$ is a single arc having an endpoint \mathbf{v}_{max} . Moreover, when H_θ does not contain v_1 or v_2^∂ , a_θ has an endpoint on either h_1 or h_2 . When H_θ does contain v_1 (respectively v_2^∂), $v_1 \subset a_\theta$ (respectively $v_2^\partial \subset a_\theta$). In particular, the braid fibration induces a *radial foliation* on Δ_d .

Recall that $\Delta_d = D_{+1} \cup N$ where the points of N have coordinates with $r \geq 1$. Near $\partial\Delta_d$ we can use N to determine the parity of the associated destabilization. The possibilities are

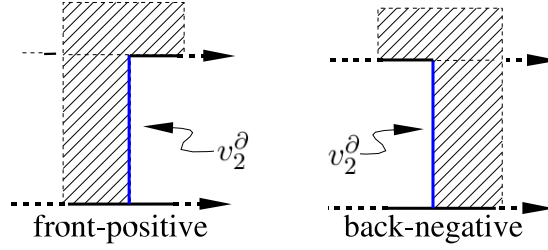


Figure 8: The illustration depicts the braid above the disc. The viewpoint is of an observer at a point with $r > 1$. The left configuration has the horizontal positions of the horizontal arcs increasing, whereas the right configuration has the horizontal positions of the horizontal arcs decreasing.

easily listed. As we traverse in the positive direction an edgepath neighborhood in $\partial\Delta_d$ of v_2^∂ , the horizontal position of the horizontal arcs adjacent to v_2^∂ is either increasing or decreasing. Then it is easily checked that our parity scheme is as follows. In N near v_2^∂ if we have $\partial\Delta_d$ with increasing horizontal position then the parity of the destabilization is positive. For decreasing horizontal position the associated destabilization is negative. Refer to Figure 8. We remind the reader that our choice of edge assignment has the ∂ -vertical arc as a front (respectively, back) edge for a positive (respectively, negative) destabilizing disc.

We will also see that this parity scheme and edge characterization can be used for the exchange move and flying discs.

Exchange move disc—(Braid presentation) Let X be a closed n -braid which admits an exchange move, i.e. the corresponding braid word $\beta(X) = WU$ where $W \in \mathcal{W}^t$, $U \in \mathcal{U}^s$ and $1 < s \leq t < n - 1$. Then there exists a *exchange disc* Δ_e above the braid having the following properties.

- E-a. $\partial\Delta_e = \alpha_{h_1} \dot{\cup} \alpha_{v_1}^\partial \dot{\cup} \alpha_{h_2} \dot{\cup} \alpha_{v_2}^\partial \dot{\cup}$ where we have the *horizontal boundary* $\alpha_{h_1}, \alpha_{h_2} \subset X$; and the *∂ -vertical arcs* $\alpha_{v_i}^\partial \subset H_{\theta_i}^\partial$ where $H_{\theta_i}^\partial \in \mathbf{H}$, $i = 1, 2$. We require that there be both a front and a back edge ∂ -vertical arc.
- E-b. $\Delta_e \cap X = \alpha_{h_1} \cup \alpha_{h_2}$.
- E-c. Δ_e transversely intersects \mathbf{A} at a single vertex point \mathbf{v}_{max} .
- E-d. $\{\alpha_{h_1} \cup \alpha_{h_2}\} \cap H_\theta \neq \emptyset$ for $H_\theta \in \mathbf{H}$ and $\theta \notin \{\theta_1^\partial, \theta_2^\partial\}$.
- E-e. $a_\theta = \Delta_e \cap H_\theta$ is a single arc having: an endpoint on \mathbf{v}_{max} ; an endpoint on $\alpha_{h_1} \cup \alpha_{h_2}$ when $\theta \notin \{\theta_1^\partial, \theta_2^\partial\}$; and contains $\alpha_{v_i}^\partial$ when $\theta = \theta_i^\partial$, $i = 1, 2$. In particular, the braid fibration induces a *radial foliation* on Δ_e .

If X' is braid isotopic to X then we can extend the braid isotopy which takes X to X' to an ambient isotopy of $S^3 \setminus \mathbf{A}$. This ambient isotopy takes Δ_e to a exchange disc for X' , i.e. properties E-a through E-e are still satisfied. Thus, every n -braid representative of $\mathcal{B}_n(X)$ will have an exchange disc. Again, the edge assignment of ∂ -vertical arcs is stable under braid isotopy.

(Arc presentation) Again with $\beta(X) = WU$, we consider the arc presentation coming from the transition $X \xrightarrow{N} X^\eta$. Then we also have a disc, which is call an *obvious exchange disc* above the braid, that has the following properties.

- EA-a. $\partial\Delta_e = h_1 \dot{\cup} v_1^\partial \dot{\cup} h_2 \dot{\cup} v_2^\partial \dot{\cup}$ where h_1 and h_2 are subarcs of two different horizontal arcs of X^η ; and $v_i^\partial \subset H_{\theta_i}^\partial$ for $H_{\theta_i}^\partial \in \mathbf{H}$, $i = 1, 2$, are the ∂ -vertical arcs of $\partial\Delta_e$. We require that there be both a front and a back edge ∂ -vertical arc.
- EA-b. $\Delta_e \cap X^\eta = h_1 \cup h_2$.
- EA-c. Δ_e transversely intersects \mathbf{A} at a single vertex point \mathbf{v}_{max} .
- EA-d. $\{h_1 \cup h_2\} \cap H_\theta \neq \emptyset$ for $H_\theta \in \mathbf{H}$ and $\theta \notin \{\theta_1^\partial, \theta_2^\partial\}$.
- EA-e. $a_\theta = \Delta_e \cap H_\theta$ is a single arc having an endpoint \mathbf{v}_{max} . Moreover, when H_θ does not contain v_i^∂ , a_θ has an endpoint on either h_1 or h_2 . When H_θ does contain v_1^∂ (respectively v_2^∂), $v_1^\partial \subset a_\theta$ (respectively $v_2^\partial \subset a_\theta$). In particular, the braid fibration induces a *radial foliation* on Δ_d .

The reader should observe that when we consider $N \subset \Delta_e$ of $\partial\Delta_e$ near v_1^∂ and v_2^∂ , we must necessarily have one horizontal boundary arc being positive and one being negative in the sense of Figure 8.

Flying disc—(Braid presentation) Let X be a closed n -braid which admits an elementary flype, i.e. $\beta(X) = W_1\sigma_{n-1}^p W_2\sigma_{n-1}^{\pm 1}$. The embedded disc we will use to illustrate the presence of a flype can conceptually be seen as an amalgamation of a destabilizing disc and an exchange disc, since the flype involves a 'flying crossing' and 'flying block'. Specifically, there exists a *flying disc* Δ_f having the following properties.

- F-a. $\partial\Delta_f = \alpha_{h_1} \dot{\cup} \alpha_{v_1}^\partial \dot{\cup} \alpha_{h_2} \dot{\cup} \alpha_{v_2}^\partial \dot{\cup} \alpha_{h_3} \dot{\cup} \alpha_{v_3}^\partial \dot{\cup}$ where we have the *horizontal boundary arcs* $\alpha_{h_i} \subset X$; and the *∂ -vertical arcs* $\alpha_{v_i}^\partial \subset H_{\theta_i}^\partial$ where $H_{\theta_i}^\partial \in \mathbf{H}$, $i = 1, 2, 3$. When Δ_f corresponds to a positive (respectively, negative) flype we have edge assignments as follows: $\alpha_{v_1}^\partial$ front, $\alpha_{v_2}^\partial$ back, $\alpha_{v_3}^\partial$ front (respectively, $\alpha_{v_1}^\partial$ back, $\alpha_{v_2}^\partial$ front, $\alpha_{v_3}^\partial$ back).
- F-b. $\Delta_f \cap X = \alpha_{h_1} \cup \alpha_{h_2} \cup \alpha_{h_3}$.
- F-c. Δ_f transversely intersects \mathbf{A} at a single vertex point \mathbf{v}_{max} .

F-d. $\{\alpha_{h_1} \cup \alpha_{h_2} \cup \alpha_{h_3}\} \cap H_\theta \neq \emptyset$ for $H_\theta \in \mathbf{H}$ and $\theta \notin \{\theta_1^\partial, \theta_2^\partial, \theta_3^\partial\}$.

F-e. $a_\theta = \Delta_f \cap H_\theta$ is a single arc having: an endpoint on \mathbf{v}_{max} ; an endpoint on $\alpha_{h_1} \cup \alpha_{h_2} \cup \alpha_{h_3}$ when $\theta \notin \{\theta_1^\partial, \theta_2^\partial, \theta_3^\partial\}$; and contains $\alpha_{v_i}^\partial$ when angle is one of the angles θ_i^∂ , $i = 1, 2, 3$. In particular, the braid fibration induces a *radial foliation* on Δ_f .

If X' is braid isotopic to X then we can extend the braid isotopy which takes X to X' to an ambient isotopy of $S^3 \setminus \mathbf{A}$. This ambient isotopy takes Δ_f to a flying disc for X' , i.e. properties F-a through F-e are still satisfied. Thus, every n -braid representative of $\mathcal{B}_n(X)$ will have an flying disc.

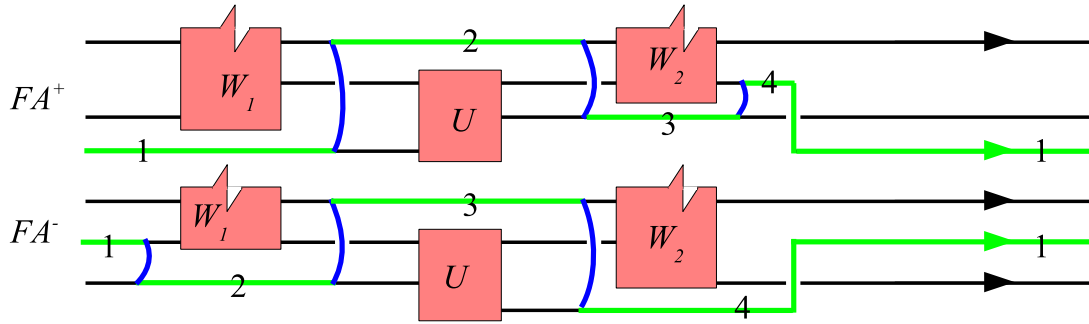


Figure 9: The figure illustrates the boundaries of the obvious flying discs when the vertical arcs and boundary arcs are above. FA^+ is the positive flype and FA^- is the negative flype. To reduce clutter we simply label the horizontal arcs h_i (in green) with their subscripts. The three ∂ -vertical arcs v_i^∂ , $i = 1, 2, 3$, are in blue and the single vertical arc, v_4 is in green.

(Arc presentation) Again with $\beta(X) = W_1 U W_2 \sigma_{n-1}^{\pm 1}$, we consider the arc presentation coming from the transition $X \xrightarrow{N} X^\eta$. Then we also have a disc, which is call an *obvious flying disc*.

FA-a. $\partial\Delta_f = h_1 \dot{\cup} v_1^\partial \dot{\cup} h_2 \dot{\cup} v_2^\partial \dot{\cup} h_3 \dot{\cup} v_3^\partial \dot{\cup} h_4 \dot{\cup} v_4 \dot{\cup}$ where h_1, h_2, h_3, h_4 are subarcs of differing horizontal arcs of X^η ; $v_i^\partial \subset H_{\theta_i^\partial}$ for $H_{\theta_i^\partial} \in \mathbf{H}$, $i = 1, 2, 3$ are ∂ -vertical arcs with $r \geq 1$; and v_4 is a vertical arc of X^η . When Δ_f corresponds to a positive (respectively, negative) flype we have edge assignments as follows: v_1^∂ front, v_2^∂ back, v_3^∂ front (respectively, v_1^∂ back, v_2^∂ front, v_3^∂ back).

FA-b. $\Delta_f \cap X = h_2 \cup h_3 \cup (h_4 \dot{\cup} v_4 \dot{\cup} h_1)$.

FA-c. $v_1^\partial, v_2^\partial \subset \Delta_f$ are above the braid with opposite parity in the sense of Figure 8.

FA-d. v_3^∂ is above and positive (respectively, negative) in which case X admits a positive (respectively, negative) flype.

FA-e. Δ_f transversely intersects \mathbf{A} at a single vertex point \mathbf{v}_{max} .

FA-f. The braid fibration induces a radial foliation on Δ_f . That is, for all $H_\theta \in \mathbf{H}$, $\Delta_f \cap H_\theta$ is a single arc having \mathbf{v}_{max} as one endpoint; when H_θ does not contain a vertical arc of X^η or a ∂ -vertical arc of $\partial\Delta_f$ then the intersection arc also has an endpoint on a horizontal portion of $\partial\Delta_f$; and, when H_θ does contain a vertical arc or ∂ -vertical arc then that arc is in the single arc of $\Delta_f \cap H_\theta$.

Returning our focus on establishing Theorem 4, we now have the following proposition.

Proposition 7 *Let X be a closed n -braid such that $\mathcal{B}_n(X)$ admits, respectively, a destabilization, exchange move or elementary flype. Consider any arc presentation coming from a presentation transition $X \xrightarrow{N} X^\eta$. Then there exists a set of intervals \mathcal{I} and a sequence of arc presentations*

$$X_{\mathcal{I}}^\eta = X^0 \rightarrow X^1 \rightarrow \cdots \rightarrow X^l = X_{\mathcal{I}}^{\prime\eta}$$

such that:

1. *If $\mathcal{B}_n(X)$ admits, respectively, a destabilization, exchange move or elementary flype then \mathcal{I} has, respectively, one, two or three intervals.*
2. *X^{i+1} is obtained from X^i via one of the elementary moves—horizontal exchange move, vertical exchange move, horizontal simplification, vertical simplification, shear horizontal exchange move, and shear vertical simplification.*
3. *$\mathcal{C}(X^{i+1}) \leq \mathcal{C}(X^i)$ for $0 \leq i \leq l$. In particular, if $X^i \rightarrow X^{i+1}$ corresponds to a horizontal, vertical or shear horizontal exchange move then $\mathcal{C}(X^i) = \mathcal{C}(X^{i+1})$. If it corresponds to a horizontal, vertical or shear vertical simplification then $\mathcal{C}(X^{i+1}) < \mathcal{C}(X^i)$. Thus, our sequence will monotonically simplify.*
4. *If $\mathcal{B}_n(X)$ admits, respectively, a destabilization, exchange move or elementary flype then here exists an obvious, respectively, destabilizing, exchange, or flying disc Δ_ε , $\varepsilon \in \{d, e, f\}$ for $X_{\mathcal{I}}^{\prime\eta}$ such that the angular positions at which the ∂ -vertical arcs of Δ_ε occur are in the set of H_θ containing $\partial\mathcal{I}$.*

The proof of Proposition 7 requires understanding “notch discs”.

2.2 Notch discs.

Let X be a closed n -braid presentation that is braid isotopic to a braid that admits either a destabilization, exchange move or elementary flype. Let Δ_ε , $\varepsilon \in \{d, e, f\}$, be an appropriate disc illustrating the isotopy. We take a transition from a braid to an arc presentation, $X \xrightarrow{N} X^\eta$,

and consider the impact of this transition on Δ_ε . As before, we denote the leaves of the radial foliation of Δ_ε by $a_\theta (= \Delta_\varepsilon \cap H_\theta)$; and a_θ contains a ∂ -vertical arc when θ is θ_i^∂ for an appropriate $i = \{1, 2, 3\}$. Since X can be positioned to be arbitrarily close to X^η (where closeness is measured by the standard metric for the (r, θ, z) coordinates), by an ambient isotopy of S^3 that preserves the boundary and foliation properties of Δ_ε , we can assume that X^η intersects Δ_ε such that we have the following:

- i. For each leaf a_θ , where $\theta \in S^1$ is not an angular position of a ∂ -vertical arc, we have that a_θ intersects either the interior of a single horizontal arc or the interior of a single vertical arc of X^η .
- ii. For a_θ where $\theta \in S^1$ is not an angular position of a ∂ -vertical arc, if a_θ intersects the interior of a horizontal arc of $h \subset X^\eta$ then $a_\theta \cap h$ is a single point.
- iii. For a_θ where $\theta \in S^1$ is not an angular position of a ∂ -vertical arc, if a_θ intersects the interior of a vertical arc of $v \subset X^\eta$ then $v \subset a_\theta$.
- iv. For a_θ containing ∂ -vertical arc v^∂ , there are two horizontal arcs $h', h'' \subset X^\eta$ such that
 - a. $a_\theta \cap X^\eta = a_\theta \cap \{h' \cup h''\}$
 - b. v^∂ has its endpoints on h' and h'' .

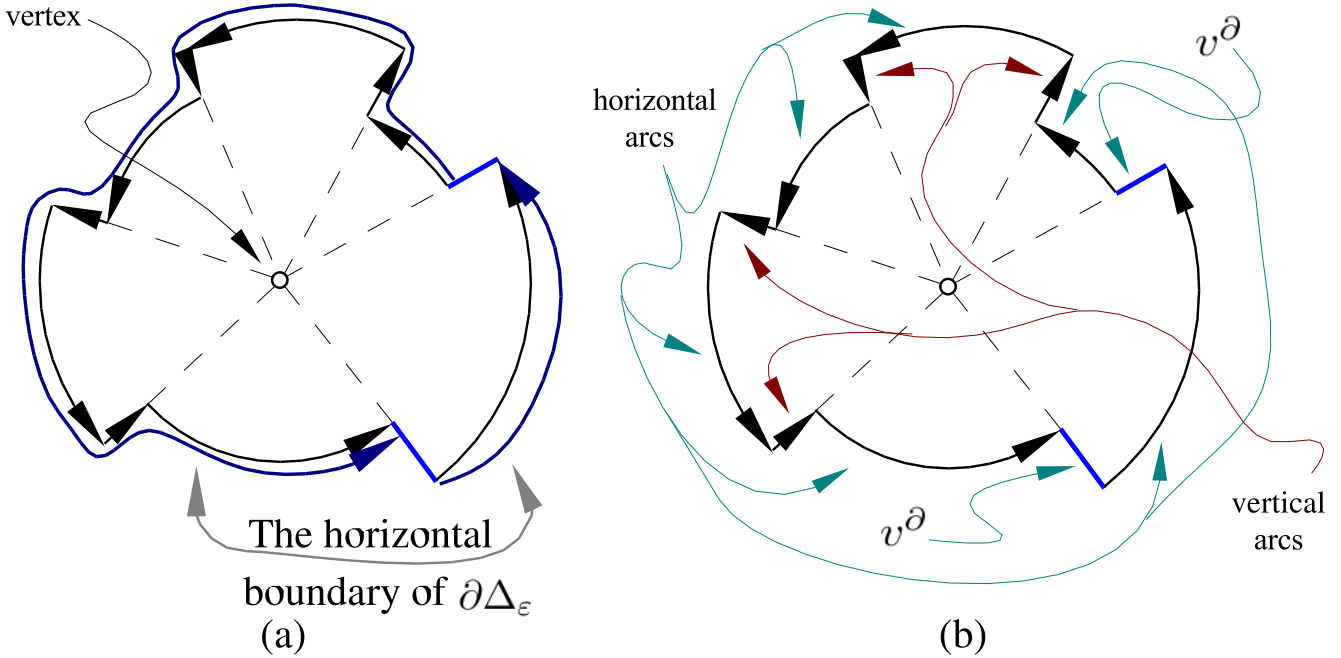


Figure 10: In illustrate (a) we have drawn $\Delta_\varepsilon^\eta \subset \Delta_\varepsilon$. In (b) we indicate what the structure of $\partial\Delta_\varepsilon^\eta$.

With these conditions holding we can *notch* Δ_ε to produce Δ_ε^η . That is, $\Delta_\varepsilon^\eta \subset \Delta_\varepsilon$ is the sub-disc whose boundary is obtain by projecting $X \cap \Delta_\varepsilon$ along the leaves a_θ to $X^\eta \cap \Delta_\varepsilon$. (See Figure 10.) The boundary of Δ_ε^η is then a union of three types of arcs: *horizontal arcs*, h_j^η , that can be either arcs or sub-arcs of the horizontal arcs of X^η ; *vertical arcs*, v_j^η , that are in fact arcs coming from the vertical arcs of X^η ; and ∂ -*vertical arcs*, v_i^∂ . We then have $\partial\Delta_\varepsilon^\eta$ being a cyclic ordered union of arcs alternating between horizontal and vertical arc types, i.e.

$$\partial\Delta_\varepsilon^\eta = h_1^\eta \dot{\cup} v_1^\eta \dot{\cup} \dots \dot{\cup} h_{j1}^\eta \dot{\cup} v_1^\partial \dot{\cup} h_{j1+1}^\eta \dot{\cup} \dots \dot{\cup} h_{jp}^\eta \dot{\cup} v_p^\partial \dot{\cup},$$

where p is 1 for $\varepsilon = d$, 2 for $\varepsilon = e$ and 3 for $\varepsilon = f$.

Extending our notation for the transition between braid presentations and arc presentations, we will use $(X, \Delta_\varepsilon) \xrightarrow{N} (X^\eta, \Delta_\varepsilon^\eta)$ and $(X^\eta, \Delta_\varepsilon^\eta) \xrightarrow{B} (X, \Delta_\varepsilon)$ for indicating the transition between presentation-disc pairs. Clearly, given a pair $(X^\eta, \Delta_\varepsilon^\eta)$ when elementary moves are applied to X^η there can be an alteration to the positioning of Δ_ε^η and/or the arc decomposition of $\partial\Delta_\varepsilon^\eta$.

Lemma 8 (First simplification of $(X^\eta, \Delta_\varepsilon^\eta)$) *Let X be a closed n -braid such that $\mathcal{B}_n(X)$ admits, respectively, a destabilization, exchange move or elementary flype and let Δ_ε be, respectively, a destabilizing disc, exchange move disc, or elementary flying disc, i.e. $\varepsilon \in \{d, e, f\}$. Then there exists an alternate disc Δ'_ε such that for $(X, \Delta'_\varepsilon) \xrightarrow{N} (X^\eta, \Delta_\varepsilon^\eta)$ we have*

- a. X^η is unchanged.
- b. $\partial\Delta_\varepsilon^\eta = h_1^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} \dots \dot{\cup} h_l^\eta \dot{\cup} v_l^\eta \dot{\cup}$ when $\varepsilon = d$.
- c. $\partial\Delta_\varepsilon^\eta = h_1^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} \dots \dot{\cup} h_l^\eta \dot{\cup} \alpha_{\theta_2}^\eta \dot{\cup}$ when $\varepsilon = e$.
- d. $\partial\Delta_\varepsilon^\eta = h_1^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} \alpha_{\theta_2}^\eta \dot{\cup} h_3^\eta \dot{\cup} v_3^\eta \dot{\cup} \dots \dot{\cup} h_4^\eta \dot{\cup} v_4^\eta \dot{\cup} \dots \dot{\cup} h_l^\eta \dot{\cup} \alpha_{\theta_3}^\eta \dot{\cup}$ when $\varepsilon = f$.

Proof. The statements b. through d. are achieved by performing an isotopy of the arcs $\alpha_{\theta_i}^\eta$'s. In particular, statement b. is true by construction. To achieve statement c. while maintaining the truth of statement a. we start with the ∂ -vertical arcs of Δ_ε^η , $\alpha_{\theta_1}^\eta, \alpha_{\theta_2}^\eta$. If these two arcs have endpoints on a common horizontal arc of X^η then we are done. If not then we push $\alpha_{\theta_2}^\eta$ backward (or forward) through the disc fibers of \mathbf{H} . This push will naturally isotop $\alpha_{\theta_2}^\eta$ in the disc fibers. We stop our push when $\alpha_{\theta_2}^\eta$ has an endpoint on a horizontal arc that $\alpha_{\theta_1}^\eta$ also has an endpoint on. This corresponds to an ambient isotopy of the graph $X^\eta \cup \alpha_{\theta_1}^\eta \cup \alpha_{\theta_2}^\eta$ in $\mathbb{R}^3 \setminus \mathbf{A}$. There is still a disc whose boundary is the union of two subarc and the resulting two ∂ -vertical arcs. This new disc is our Δ'_ε . It is easy to see that Δ'_ε is in fact a notch disc.

Similarly, for achieving statement d. while maintaining the truth of statement a. we first push $\alpha_{\theta_2}^\eta$ backward until it has an endpoint on a horizontal arc that also contains an endpoint of $\alpha_{\theta_1}^\eta$. Not we push $\alpha_{\theta_3}^\eta$ backward until it has an endpoint on a horizontal arc that contains

an endpoint of $\alpha_{\theta_2}^\eta$. Since both pushes are ambient isotopies of $\mathbb{R}^3 \setminus \mathbf{A}$ we again have an new notch disc. \diamond

We will refer to the portion $h_2^\eta \cup v_2^\eta \cup \dots \cup h_l^\eta \cup v_l^\eta \subset \partial\Delta_d^\eta$ as the *middle boundary* of Δ_d^η . The *middle boundary* of $\partial\Delta_e^\eta$ (respectively $\partial\Delta_f^\eta$) is $h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} \dots \dot{\cup} h_l^\eta$ (respectively, $h_3^\eta \dot{\cup} v_3^\eta \dot{\cup} \dots \dot{\cup} h_4^\eta \dot{\cup} v_4^\eta \dot{\cup} \dots \dot{\cup} h_l^\eta$). Proposition 7 and, thus, Theorem 4 will be established when the middle boundary of Δ_ε^η has been simplified so that it contains only a single horizontal boundary arc. Our edge assignment assumptions give us that middle boundary edgepath for $\partial\Delta_e^\eta$ start on back edge ∂ -vertical arc and ends on a front edge ∂ -vertical arc. Whereas, for a positive (respectively, negative) flyping disc $\partial\Delta_e^\eta$ the middle boundary edgepath starts and ends on a front edge (respectively, back edge) ∂ -vertical arc.

Recalling our decomposition $\Delta_\varepsilon = D_{+1} \cup N$, the reader should notice that the notching transition and the argument of Lemma 8 alters only the annulus N and leaves D_{+1} untouched. In particular, the plane $z = z_{max}$ containing D_{+1} still has z_{max} being greater than the horizontal position of all horizontal arcs of X^η . Abusing notation we will have the decomposition $\Delta_\varepsilon^\eta = D_{+1} \cup N$.

For the remainder of our discussion we will assume that our notch disc satisfies the conclusion of Lemma 8.

2.3 The intersection of C_1 & Δ_ε^η .

In this subsection we start with a given initial pair $(X^\eta, \Delta_\varepsilon^\eta)$ and analyze the intersection $C_1 \cap \Delta_\varepsilon^\eta$. Our overall strategy is to simplify $C_1 \cap \Delta_\varepsilon^\eta$ until Δ_ε^η is an obvious disc illustrating either a destabilization, exchange move or elementary flype.

We consider the intersection $C_1 \cap \Delta_\varepsilon^\eta$. Notice that for each horizontal arc $h_j^\eta \subset \partial\Delta_\varepsilon^\eta$ we necessarily have $h_j^\eta \subset C_1$; for each vertical arc $v_j^\eta \subset \partial\Delta_\varepsilon^\eta$ we have $v_j^\eta \cap C_1 = \partial v_j^\eta$; and we can assume that the interior of each vertical boundary arc v^∂ transversally intersects C_1 at finitely many points. We can assume that C_1 and $\text{int}(\Delta_\varepsilon^\eta)$ intersect transversely. Thus, $C_1 \cap \text{int}(\Delta_\varepsilon^\eta)$ is a union of simple arcs (**sa**) and simple closed curves (**scc**).

From our decomposition $\Delta_\varepsilon^\eta = D_{+1} \cup N$ we know that we have a distinguished **scc** $\partial D_{+1} \subset C_1 \cap \Delta_\varepsilon^\eta$ which we will denote by c_{max} . All other **scc** and **sa** of $C_1 \cap \Delta_\varepsilon^\eta$ are intersections in $C_1 \cap N$.

This next lemma allows us to get some initial control over the behavior of $C_1 \cap \Delta_\varepsilon^\eta$ without altering X^η .

Lemma 9 (Second simplification of $(X^\eta, \Delta_\varepsilon^\eta)$) *Let (X, Δ_ε) be a braid presentation/disc pair where $\varepsilon \in \{d, e, f\}$, and consider an arc presentation/disc pair coming from the transition*

$(X, \Delta_\varepsilon) \xrightarrow{N} (X^\eta, \Delta_\varepsilon^\eta)$. Then we can replace the pair $(X^\eta, \Delta_\varepsilon^\eta)$ with $(X^\eta, \Delta_\varepsilon'^\eta)$ such that no **sa** has an endpoint on $X^\eta \cap \partial\Delta_\varepsilon'^\eta$. In particular, all **sa** of $\Delta_\varepsilon'^\eta \cap C_1$ have their endpoints on the ∂ -vertical arcs of $\partial\Delta_\varepsilon'^\eta$.

Proof. By an isotopy of a collar neighborhood of $\partial\Delta_\varepsilon^\eta$ in $N(\subset \Delta_\varepsilon^\eta)$ we can assume that there is a neighborhood $\mathbf{n} \subset N \setminus c_{\max}$ which has the structure $(X^\eta \cap \partial\Delta_\varepsilon'^\eta) \times I$ such that $\mathbf{n} \cap C_1 = X^\eta \cap \partial\Delta_\varepsilon'^\eta$. After this isotopy the only place where any **sa** can have its endpoints is on the v^∂ ∂ -vertical arcs. \diamond

After Lemma 9 the reader should notice that we can assume $\overline{C_1 \cap \text{int}(\Delta_\varepsilon^\eta)} = C_1 \cap (\Delta_\varepsilon^\eta \setminus X^\eta)$.

Next, we set $\mathcal{T}_0 = \{(r, \theta, z) | r < 1\}$ and $\mathcal{T}_\infty = \{(r, \theta, z) | r > 1\}$. Let $R \subset \Delta_\varepsilon^\eta \setminus (C_1 \cap \Delta_\varepsilon^\eta)$ be any component. If $R \subset \mathcal{T}_0$ (respectively $R \subset \mathcal{T}_\infty$) then we assign R a “0” (respectively “ ∞ ”) label, i.e. R^0 (respectively R^∞).

Lemma 10 (Initial position of $C_1 \cap \Delta_\varepsilon^\eta$ -part 1.) *Let (X, Δ_ε) be a braid presentation/disc pair where $\varepsilon \in \{d, e, f\}$, and consider an arc presentation/disc pair coming from the transition $(X, \Delta_\varepsilon) \xrightarrow{N} (X^\eta, \Delta_\varepsilon^\eta)$. Then we can replace the pair $(X^\eta, \Delta_\varepsilon^\eta)$ with $(X^\eta, \Delta_\varepsilon'^\eta)$ such that the following hold:*

- a. Every **scc** of $C_1 \cap \Delta_\varepsilon'^\eta$ bounds a subdisc of $\Delta_\varepsilon'^\eta$ whose associated label is 0.
- b. For every **scc** of $c \subset C_1 \cap \Delta_\varepsilon'^\eta$ either c bounds a subdisc of C_1 which contains a single horizontal arc of X^η , or $c = c_{\max}$ and bounds D_{+1} .
- c. For every **sa** of $C_1 \cap \Delta_\varepsilon'^\eta$ having both endpoints on the same ∂ -vertical arc $v^\partial \subset H_{\theta^\partial}$, it is outer-most in $\Delta_\varepsilon'^\eta$ and splits off a subdisc of $\Delta_\varepsilon'^\eta$ whose associated label is 0.
- d. For every **sa** $\gamma \subset C_1 \cap \Delta_\varepsilon'^\eta$ that has both endpoints on the same ∂ -vertical arc v^∂ , there exists a sub-arc $\gamma' \subset H_{\theta^\partial} \cap C_1$ with $\partial\gamma = \partial\gamma'$ such that the bounded disc components of $C_1 \setminus [\gamma \cup \gamma'] (\subset C_1)$ intersects exactly one horizontal arc of X^η . (Note: $\gamma \dot{\cup} \gamma' \dot{\cup}$ is not necessarily a **scc**, thus there may be more than one disc component of $C_1 \setminus (\gamma \cup \gamma')$ with only one intersecting X^η .)

Proof. Our argument for all four statements involves understanding the behavior of $C_1 \cap \Delta_\varepsilon^\eta$ in the radial foliation of Δ_ε^η . To start, we assume that all but finitely many points in the components of $C_1 \cap \Delta_\varepsilon^\eta$ are transverse to the radial foliation of Δ_ε^η . Moreover, we assume that the points tangency are generic (local max or min) and each leaf in the radial foliation has at most one point of tangency.

We first deal have a *simple situation*. Let $c \subset C_1 \cap \Delta_\varepsilon^\eta$ be a **scc** such that: c is innermost on both C_1 and Δ_ε^η ; c bounds a disc $R^\infty \subset \Delta_\varepsilon^\eta$; and, c is tangent at exactly two points to the

radial foliation of Δ_ε^η . Then R^∞ splits off a 3-ball in \mathcal{T}_∞ whose interior has empty intersection with X^η . We can isotop R^∞ through this 3-ball so as to eliminate c and reducing $|C_1 \cap \Delta_\varepsilon^\eta|$.

Next, let $p \in C_1 \cap \Delta_\varepsilon^\eta$ be a point of tangency. Suppose there exists a closed subarc in the leaf of the radial foliation $\gamma \subset \Delta_\varepsilon^\eta \cap H_\theta$ such that: $p \in \gamma$; $\partial\gamma \subset C_1 \cap \Delta_\varepsilon^\eta$ (we allow for $p \in \partial\gamma$); and, $(\gamma \setminus p) \cap \mathcal{T}_\infty = \text{int}(\gamma) \cap \mathcal{T}_\infty (\neq \emptyset)$. Then we say p is a *extraneous tangency*. (We observe that in our simple situation of c being a **scc** having exactly two points of tangency, neither tangency was extraneous.)

Let E be the number of extraneous tangency of $C_1 \cap \Delta_\varepsilon^\eta$. Our immediate goal is to show how we can reduce E to zero. To do this we first observe that there are two types of tangent points in $C_1 \cap \Delta_\varepsilon^\eta \subset \Delta_\varepsilon^\eta$. For a tangent point $p \in C_1 \cap \Delta_\varepsilon^\eta$ let γ , again, be a small enough subarc of the leaf in the radial foliation of Δ_ε^η such that: $p \in \gamma$ and $(\gamma \setminus p) \cap C_1 = \emptyset$. Then either [type-1] $\gamma \cap \mathcal{T}_0 \neq \emptyset$ or [type-2] $\gamma \cap \mathcal{T}_\infty \neq \emptyset$.

It is the type-1 where $\gamma \cap \mathcal{T}_0 \neq \emptyset$ that creates the possibility of an extraneous tangency. One can see this as follows. Let $c \subset C_1 \cap \Delta_\varepsilon^\eta$ be the **scc** or **sa** component with $p \in c$. Without loss of generality suppose that as we push γ forward in the radial foliation of Δ_ε^η it becomes a “secant” in \mathcal{T}_∞ near the tangency p , intersecting c twice. (Pushing γ in the backward direction would move γ off of c .) Now thinking of γ as this newly formed secant it will have its two endpoints on c and it will intersect \mathcal{T}_∞ . If we continue to push our secant γ forward (maintaining the feature that its endpoints are sliding along c) we stop when one of three events occurs: 1) γ encounters another tangent point $p' \in C_1 \cap \Delta_\varepsilon^\eta$; 2) γ encounters a ∂ -vertical arc; or 3) γ shrinks down and becomes a tangent point. If event-3 occurs then c was a **scc** meeting the assumptions of our simple situation. For the moment we allow for the occurrence of event-2. If event-1 occurs then the new tangent point γ encounters is extraneous. Moreover, the configuration in event-1 satisfies the following features.

- a. $B \subset \mathcal{T}_\infty$ is an open 3-ball with $\overline{B} \cap X^\eta = \emptyset$.
- b. $\partial\overline{B}$ is $\overline{R_1 \cup R_2 \cup R_3}$.
- c. $R_1 \subset \Delta_\varepsilon^\eta \cap \mathcal{T}_\infty$ is a subdisc with $\partial\overline{R_1} = \gamma \dot{\cup} \alpha \dot{\cup}$ where:
 - i. $\gamma \subset \Delta_\varepsilon^\eta \cap H_{\theta_0}$ is an event-1 secant containing the extraneous tangent point p' ;
 - ii. $\alpha \subset c$ with $p \in \alpha$;
- d. $R_2 \subset C_1$ is a subdisc with $\partial\overline{R_2} = \alpha \dot{\cup} \alpha' \dot{\cup}$ where α is from c-ii and $\alpha' \subset C_1$ is a subarc of $C_1 \cap H_{\theta_0}$.
- e. $R_3 \subset H_{\theta_0} \cap \mathcal{T}_\infty$ is a subregion with $\partial\overline{R_3} = \gamma \dot{\cup} \alpha' \dot{\cup}$. (Depending on whether p is in $\partial\gamma$ or $\text{int}(\gamma)$, R_3 is either a subdisc or the wedge of two subdiscs with p' as the wedge point.)

We are now in a position we see how E can be reduced to zero. We consider the disc $R_3 \subset H_{\theta_0}$. Since $\gamma \subset \overline{R_3}$, see that $p' \subset \overline{R_3}$. Assuming that $R_3 \cap \Delta_\varepsilon^\eta = \emptyset$, in a product neighborhood of R_3 we can push γ into \mathcal{T}_0 dragging Δ_ε^η along to eliminate p' as an extraneous tangent point. (The key feature of this isotopy is that the radial foliation of Δ_ε^η is unchanged.) For the new Δ_ε^η , $C_1 \cap \Delta_\varepsilon^\eta$ will have a new simple situation **scc**. If p' had been a boundary endpoint of γ then the number of tangent points in $C_1 \cap \Delta_\varepsilon^\eta$ is the same. If p' had been in $\text{int}(\gamma)$ then $C_1 \cap \Delta_\varepsilon^\eta$ has two additional tangent points in the radial foliation. However, E has gone down by one.

For $R_3 \cap \Delta_\varepsilon^\eta \neq \emptyset$ we recall that p' is the only tangent point in H_{θ_0} . So $R_3 \cap \Delta_\varepsilon^\eta$ is a collection of **sa**. Starting with an outermost such **sa**, in a product neighborhood of R_3 we push these **sa**'s into \mathcal{T}_0 , dragging Δ_ε^η along. It is readily seen that no new extraneous tangent points are introduced. Iterating this procedure we can assume $E = 0$. Moreover, after eliminating simple situation **scc** we can assume statement-a is true.

To establish statement b. let $c \subset C_1 \cap \Delta_\varepsilon^\eta$. Assume that c bounds a sub-disc $R \subset \Delta_\varepsilon^\eta$. Now that we have established the validity of statement a. we can assume that $\text{int}(R) \subset \mathcal{T}_0$. If $R \cap \mathbf{A} \neq \emptyset$ then R is in fact D_{+1} and we have $c = c_{\max}$. Moreover, c_{\max} is unique in having the feature that it bounds a disc which intersects \mathbf{A} .

Now assume $R \cap \mathbf{A} = \emptyset$. Thus, c must bound on C_1 . Let $\Delta_c \subset C_1$ having $\partial\Delta_c = c$. Moreover, c contains exactly two tangent points for otherwise c would contain an extraneous tangent point. This allows us to assume that we can isotop R , first, arbitrarily close to C_1 , then onto a subdisc of C_1 by radially pushing R out of \mathcal{T}_0 along θ -rays. The obstruction to pushing R totally out of \mathcal{T}_0 will be horizontal arcs of X^η that are contained in Δ_c . Thus, this radially isotopy of R will result in a new **scc** for each horizontal arc of $\Delta_c \cap X^\eta$.

The arguments establishing statements c. & d. are similar. For statement c. let $\alpha \subset C_1 \cap \Delta_\varepsilon^\eta$ be a **sa** that splits off a region $R \subset \Delta_\varepsilon^\eta$. Assume that $\partial\alpha$ is contained in the ∂ -vertical arc $v^\partial \subset H_{\theta^\partial}$ where $H_{\theta^\partial} \in \mathbf{H}$. The first two possibilities to consider is either $R \cap \mathcal{T}_\infty = \emptyset$ or $R \cap \mathcal{T}_\infty \neq \emptyset$.

Suppose $R \cap \mathcal{T}_\infty \neq \emptyset$ and assume that $E = 0$ and all simple situation **scc**'s have been eliminated. Then α contains either one or three tangent points with the radial foliation of Δ_ε^η . The reader should readily see that the arc $v^\partial \cap R (\subset H_{\theta^\partial})$ splits off a subdisc of $\mathcal{T}_\infty \cap H_{\theta^\partial}$ that does not intersect X^η . Thus, we can push this arc into \mathcal{T}_0 reducing $|C_1 \cap v^\partial|$. We eliminate any extraneous tangencies or simple situation **scc** that are produced. Ultimately, this will result in pushing the R -portion of Δ_ε^η into \mathcal{T}_0 thus reducing $|C_1 \cap \Delta_\varepsilon^\eta|$.

Now suppose $R \cap \mathcal{T}_0 = \emptyset$ and assume that $E = 0$. It is easily seen that α contains either [case-1] one or [case-2] three points of tangencies. For case-1 the one point of tangency must be a type-2. For case-2 two of the tangencies must be type-2 and one a type-1.

Considering case-1, α has exactly one point of tangency with the radial foliation of Δ_ε^η . To be descriptive, the angular position of $\gamma(= R \cap v^\partial)$ is an endpoint of the interval that is the angular support of α ; and, the angular position of the type-2 point of tangency is the other endpoint of the angular support of α . The vertical support of α and γ are equal.

Recall that $\gamma \subset v^\partial \subset H_{\theta\partial}$. Let $\gamma' \subset C_1 \cap H_{\theta\partial}$ such that $\gamma' \dot{\cup} \alpha \dot{\cup}$ bounds a subdisc $R' \subset C_1$. In \mathcal{T}_0 we can isotop R close enough to C_1 such that by radially pushing R out of \mathcal{T}_0 along θ -rays R is bijectively mapped onto R' . The obstruction to pushing R totally out of \mathcal{T}_0 will be horizontal arcs of X^η that are contained in R' . Thus, this radially isotopy of R will result in a new **sa**'s for each horizontal arc of $R' \cap X^\eta$. Thus, for case-1 **sa**'s we obtain statements c. and d.

Statements c. and d. for case-2 **sa** is achieve in almost similar fashion. Notice that with $E = 0$, our arc α will have exactly three points of tangency. Being descriptive, we can traverse α so that we have, in order points of tangency, $p_1, p_2, p_3 \subset \alpha$ where p_1 is a type-1 tangency that leads to an event-2 secant-push. The angular position of γ & γ' is in the interior of the interval that is the angular support of α . And, the vertical support of α properly contains the vertical support of γ & γ' . (For measures of vertical and angular support it is best to visualize these support projected onto C_1 .) We can move the angular position of p_1 and p_2 arbitrarily close to that of γ & γ' thus making the vertical support of α arbitrarily close to that of γ & γ' . (Due to the fact that we are maintaining the constant nature of the edge assignment of v^∂ , the geometry of R is creating a “fold” near γ .)

The closed curve $\gamma' \dot{\cup} \alpha \dot{\cup}$ is either simple or has two points of self intersection. Either way $C_1 \setminus [\gamma' \dot{\cup} \alpha \dot{\cup}]$ has bounded disc components. In \mathcal{T}_0 we can isotop R close enough to C_1 by radially pushing R out of \mathcal{T}_0 along θ -rays. (The image of R onto C_1 under this radial push will always be more than a disc region due to the previously mentioned fold.) The obstruction to pushing R and γ totally out of \mathcal{T}_0 will be horizontal arcs of X^η that are contained in the bounded disc components of $C_1 \setminus [\gamma' \dot{\cup} \alpha \dot{\cup}]$. Thus, this radially isotopy of R will result in a new **sa**'s for each horizontal arc of $R' \cap X^\eta$. Thus, for case-1 **sa**'s we obtain statements c. and d. \diamond

Statements c & d of Lemma 10 deal with just one type of **sa** which we will be interested in. In particular, there are four types of **sa**'s in our argument. All will occur in angular intervals that is less than 2π . (See Figure 11.)

type-1 $\gamma \subset C_1 \cap \Delta_\varepsilon^\eta$ is a **sa** having both endpoints on the same ∂ -vertical arc $v^\partial \subset \partial\Delta_\varepsilon^\eta$. From the proof of Lemma 10 we know we have case-1 (one point of tangency) & case-2 (three points of tangency) for γ .

type-2 $\gamma \subset C_1 \cap \Delta_\varepsilon^\eta$ is a **sa** having its endpoints on different ∂ -vertical arc $v^\partial, v'^\partial \subset \partial\Delta_\varepsilon^\eta$. Moreover, γ is transverse to the leaves of the radial foliation of Δ_ε^η . Thus, the edge

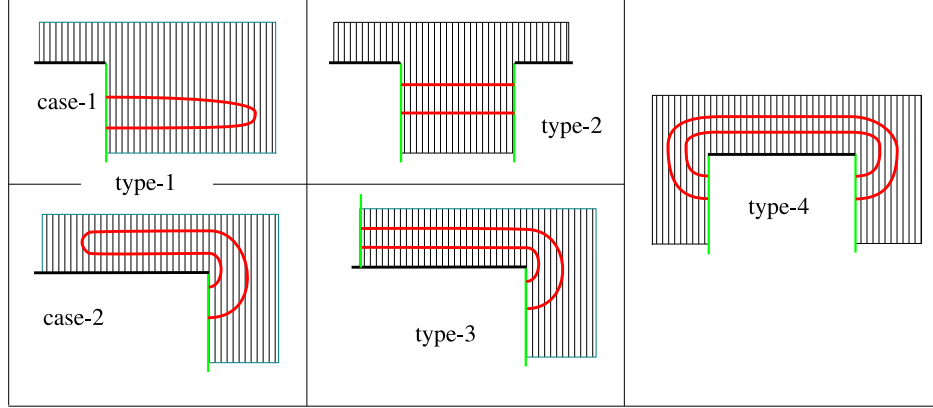


Figure 11: The vertical cross-hatching illustrates the local leaves of the radial foliation. The γ sa's are the red arcs.

assignments of v^∂ and v'^∂ must be different.

type-3 $\gamma \subset C_1 \cap \Delta_\varepsilon^\eta$ is a sa having its endpoints on different ∂ -vertical arc $v^\partial, v'^\partial \subset \partial\Delta_\varepsilon^\eta$. Moreover, γ has one point where it is not transverse to the radial foliation of Δ_ε^η . Thus, the edge assignments of v^∂ and v'^∂ must be the same.

type-4 $\gamma \subset C_1 \cap \Delta_\varepsilon^\eta$ is a sa having its endpoints on different ∂ -vertical arc $v^\partial, v'^\partial \subset \partial\Delta_\varepsilon^\eta$. Moreover, γ has two points where it is not transverse to the radial foliation of Δ_ε^η with γ transversely intersecting any leaf at most twice. Thus, the edge assignments of v^∂ and v'^∂ must be different.

Lemma 11 (Initial position of $C_1 \cap \Delta_\varepsilon^\eta$ -part 2.) *Let (X, Δ_ε) be a braid presentation/disc pair where $\varepsilon \in \{d, e, f\}$, and consider an arc presentation/disc pair coming from the transition $(X, \Delta_\varepsilon) \xrightarrow{N} (X^\eta, \Delta_\varepsilon^\eta)$. We can replace the pair $(X^\eta, \Delta_\varepsilon^\eta)$ with $(X^\eta, \Delta_\varepsilon^\eta)$ such that for every component of $\delta \subset \Delta_\varepsilon^\eta \setminus (C_1 \cap \Delta_\varepsilon^\eta)$ with label 0, if $\delta \cap \mathbf{A} = \emptyset$ and $\delta \cap X^\eta = \emptyset$ then δ is a disc of the one of the following type:*

- A whole disc–A subdisc whose boundary in $C_1 \cap \Delta_\varepsilon^\eta$ is a scc .
- A half disc–A subdisc whose boundary is the cyclic ordered union of two arcs $\gamma_1 \dot{\cup} \gamma_2 \dot{\cup}$ where $\gamma_1 \subset C_1 \cap \Delta_\varepsilon^\eta$ is a type-1 sa ; and, $\gamma_2 \subset v^\partial$ for some some ∂ -vertical arc v^∂ .
- a rectangle–A subdisc those boundary is the cyclic ordered union of four arcs $\gamma_1 \dot{\cup} \gamma_2 \dot{\cup} \gamma_3 \dot{\cup} \gamma_4 \dot{\cup}$ where: $\gamma_1, \gamma_3 \subset C_1 \cap \Delta_\varepsilon^\eta$; $\gamma_1 \not\dot{\cup} \gamma_3$ are sa's of the same type- j in Δ_ε^η , $j \in \{2, 3, 4\}$; $\gamma_2 \subset v_1^\partial$ and $\gamma_4 \subset v_2^\partial$ where $v_1^\partial(\subset H_{\theta_1}) \not\dot{\cup} v_2^\partial(\subset H_{\theta_2})$ are two different ∂ -vertical arcs. Moreover, there exists a unique horizontal arc $h \subset X^\eta$ whose angular support contains the angular interval over which δ occurs; and, whose horizontal position is contained in the horizontal interval over which δ occurs.

Proof. Consider a component $\delta^0 \subset \Delta_\varepsilon^\eta \cap \mathcal{T}_0$ where $\delta^0 \cap \mathbf{A} = \emptyset$. If $|\partial\overline{\delta^0} \cap C_1| = 1$ then either $\partial\overline{\delta^0} \cap C_1$ is a **scc** or a **sa**. If it is a **scc** then by statements a & b of Lemma 10 we have our whole-disc-statement a. If it is a **sa** then this **sa** must be a case-1 or -2 type-1 **sa** and by statements c & d of Lemma 10 we have our half-disc-statement b.

More generally, if $|\partial\overline{\delta^0} \cap C_1| > 1$ then, again, by statements a & c of Lemma 10, δ^0 is a disc planar region of Δ_ε^η . Moreover, since we can assume there are no extraneous tangencies, we conclude that $\partial\overline{\delta^0} \cap C_1$ is a collection of type-2, -3 & -4 **sa** 's. Also, since there are no extraneous tangencies we can place δ^0 arbitrarily close to C_1 so that any radial ray, $\{(r, 0, 0) | r \geq 0\}$, intersects δ^0 at most once. Thus, we can consider the attempt to radially push δ^0 out of \mathcal{T}_0 . The obstructions to pushing all of δ^0 out of \mathcal{T}_0 are any horizontal arcs of X^η that intersect the radial projection $\pi(\overline{\delta^0}) \subset C_1$. We list the possibilities. If $h \subset X^\eta \cap \pi(\overline{\delta^0})$ is a horizontal arc in the interior of $\pi(\overline{\delta^0})$ then h obstructs a whole disc in the radial push of δ^0 out of \mathcal{T}_0 . If $h \subset X^\eta$ is a portion of a horizontal arc that intersects $\partial\pi(\overline{\delta^0})$ once then h obstructs a half disc in the radial push of δ^0 out of \mathcal{T}_0 . If $h \subset X^\eta$ is a portion of a horizontal arc that intersects $\partial\pi(\overline{\delta^0})$ twice then h obstructs a rectangle in the radial push of δ^0 out of \mathcal{T}_0 . (See Figure 11 for illustrations of the three types of rectangles.) \diamond

In Lemmas 9, 10 and 11 for the pair $(X^\eta, \Delta_\varepsilon^\eta)$ we did not alter X^η , only Δ_ε^η . Thus, our complexity measure $\mathcal{C}(X^\eta)$ remained constant. The resulting pair $(X^\eta, \Delta_\varepsilon'^\eta)$ coming from the application of Lemmas 10 and 11 will be referred to as an *initial position* for the braid presentation/disc pair.

Remark 12 In §2.4 we will develop the machinery for eliminating the whole discs and half discs of Lemma 11. The reader should notice that rectangle disc of statement c. of Lemma 11 only occur for Δ_e and Δ_f discs. It is possible to replace (X, Δ_ε) and a corresponding $(X, \Delta_\varepsilon) \xrightarrow{N} (X^\eta, \Delta_\varepsilon^\eta)$, $\varepsilon \in \{e, f\}$, with a new $(X^\eta, \Delta_\varepsilon'^\eta)$ such that no component of $\Delta_\varepsilon'^\eta \cap \mathcal{T}_0$ is a rectangle subdisc. The replacement will still characterize the occurrence of an exchange move or flype. However, in line with our discussion in Remark 1, this may not be the same exchange move or flype as characterized by the original Δ_ε^η . Thus, in order to establish Theorem 6 we will need to maintain the integrity of our Δ_e and Δ_f discs.

Finally, consider γ_1 & γ_3 of statement c. having endpoints on ∂ -vertical arcs v_1^∂ and v_2^∂ , where θ_1 and θ_2 are their angular positions, respectively. Then the angular interval over which γ_1 & γ_3 occur contains, say, $[\theta_1, \theta_2]$. (The other possibility is $[\theta_2, \theta_1]$.) By a slight isotopy we can assume that the γ_1, γ_3 have constant z -coordinates over $[\theta_1, \theta_2]$ and we will refer to these constant coordinates as the horizontal positions of γ_1 & γ_3 . We will refer to them as *horizontal boundary arcs* and γ_2, γ_4 as *∂ -vertical arcs* of the rectangle. Since $[\theta_1, \theta_2]$ will be arbitrarily close to the angular interval over which γ_1 & γ_3 occur (for type-2 it is equal) we will refer to it as the angular support of these **sa** 's and the associated rectangle δ . Similarly, the vertical

interval over which δ occurs is arbitrarily close to the interval between the horizontal positions of γ_1 and γ_3 . We will refer to this interval as the vertical support of δ . We will let $\mathcal{R} \subset \Delta_\varepsilon^\eta$ be the set of all rectangle subdiscs. \diamond

2.4 The tiling machinery on Δ_ε^η .

We now make the transition from an arc presentation to a shear presentation. To do this we need to specify the intervals for \mathcal{I} . Each ∂ -vertical arc of Δ_ε will contribute one component to \mathcal{I} . So let $v_k^\partial \subset \partial\Delta_\varepsilon$ be a ∂ -vertical arc having angular position ϑ_k . Then \mathcal{I} contains an angle interval $[\vartheta_k - \epsilon_k, \vartheta_k + \epsilon_k]$. (ϵ_k is small enough so that the components of \mathcal{I} contains no vertical arcs and are pairwise disjoint; and, $1 \leq k \leq p$ where p is 1, 2 or 3 depending upon whether ε is d , e or f , respectively.) With this assignment for \mathcal{I} in place we introduce new transition notation $(X, \Delta_\varepsilon) \xrightarrow{S} (X_\mathcal{I}^\eta, \Delta_\varepsilon^\eta)$ for going from a braid/disc pair to a shear-arc-presentation/notch-disc pair.

We now adapt the classical foliation/tiling machinery of surfaces in braid structures that has been extensively developed and exploited in [BM1, BM2, BM3, BM4, BF]. This machinery has built into it the ability to recognize when and where exchange moves are admitted. It is important to notice that we will be creating a tiling only on the components of $\Delta_\varepsilon^\eta \setminus \text{int}(\mathcal{R})$. To start our adaptation we have the following definitions.

Given a pair $(X_\mathcal{I}^\eta, \Delta_\varepsilon^\eta)$ in initial position (as specified by the conclusions of Lemma 10 & 11), let $h \subset X_\mathcal{I}^\eta$ be a horizontal arc (respectively, portion of a horizontal arc) that corresponds to a horizontal arc (respectively, portion of a horizontal arc) associated with a whole disc as described in statement b. (respectively, half disc as described in statement c.) of Lemma 10.

Let $[\theta_1^h, \theta_2^h]$ be the angular support of h and z^h be the vertical position of h . The *coning disc* of h is the disc $\delta^h = \{(r, \theta, z^h) | 0 \leq r \leq 1, \theta_1^h \leq \theta \leq \theta_2^h\}$. Let $\mathbf{v}^h = \delta^h \cap \mathbf{A}$ is the vertex for δ^h . The reader should notice that a coning disc inherits a radial foliation from its intersection with the disc fibers of \mathbf{H} .

Now for any coning disc δ^h we consider the intersection set $\Delta_\varepsilon^\eta \cap \delta^h$. By the four Lemmas used to define initial position we know that this intersection set will be a union of arcs that are transverse to disc fibers of \mathbf{H} and, thus, the leaves of the radial foliation of δ^h . So viewed in δ^h , any arc of intersection with Δ_ε^η will be seen as parallel to $h \subset \partial\delta^h$. In particular, by a slight isotopy of the whole discs and half discs of $\Delta_\varepsilon^\eta \cap \mathcal{T}_0$, we can assume that the cylindrical coordinates of any intersection arc in the set $\Delta_\varepsilon^\eta \cap \delta^h$ has constant r -coordinate along with constant z -coordinate being z^h .

Let $\gamma \subset \Delta_\varepsilon^\eta \cap \delta^h$ be the intersection arc in δ^h having the smallest r -coordinate which we call r_γ . We can then isotop γ through δ^h and past the axis \mathbf{A} by letting r_γ go to zero and past \mathbf{A} .

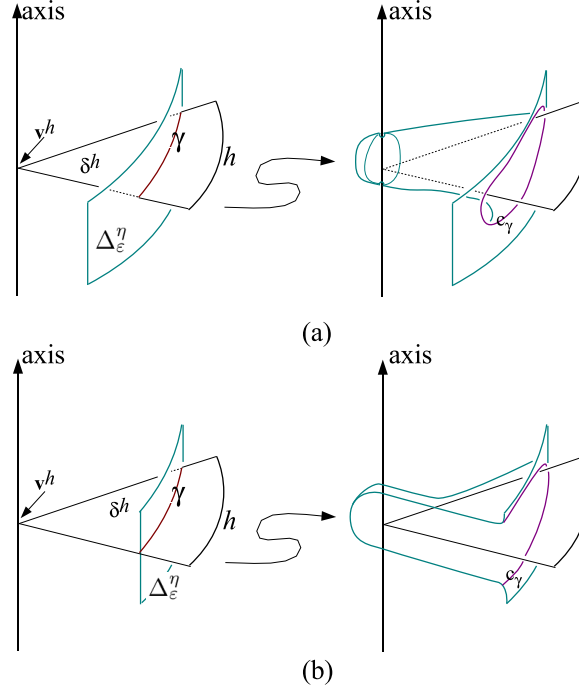


Figure 12: The corresponding alteration in the foliation in (a) is depicted in Figure 13(a). Similarly, there is a correspondence between illustration (b) and Figure 13(b). The curve c_γ is used to designate the grouping of the vertices and singularities introduced by the tiling isotopy that is associated with the intersection arc γ .

Extending this isotopy of γ to Δ_ε^η we produce a disc Δ_ε^η that has a *tilted foliation*.

As illustrated in Figure 12-right, there are two possibilities: c_γ is a **scc** or c_γ is a **sa**. However, since there are case-1 and case-2 **sa**'s, the type-1 **sa** situation further bifurcates into two slightly differing alterations to the foliation. In Figure 12 we illustrate the *tiling isotopy* for a coning disc δ^h associated with a whole disc (Figure 12(a)) and a half disc split off by a case-1 type-1 **sa** (Figure 12(b)).

As mentioned before, these tiling isotopies will occur away from the rectangle discs (statement c. Lemma 11) of our initial position. In Figure 12(a) we have the case where h is the horizontal arc associated with a whole disc. Here the tiling isotopy introduces two intersection points or *vertices* of Δ_ε^η with **A**; and two points of tangency with disc fibers of **H** or *singular points*. Turning to Figure 13(a) we see how the tiling isotopy alters the radial foliation in a neighborhood of such an intersection arc γ .

The situation for half discs is similar but a little more complex since the associated **sa** splitting off the half disc in Δ_ε^η can be either a case-1 or case-2, type-1. Taking the case-1 first, in Figure 12(b)-left we illustrate a portion, h , of a horizontal arc that is associated with said

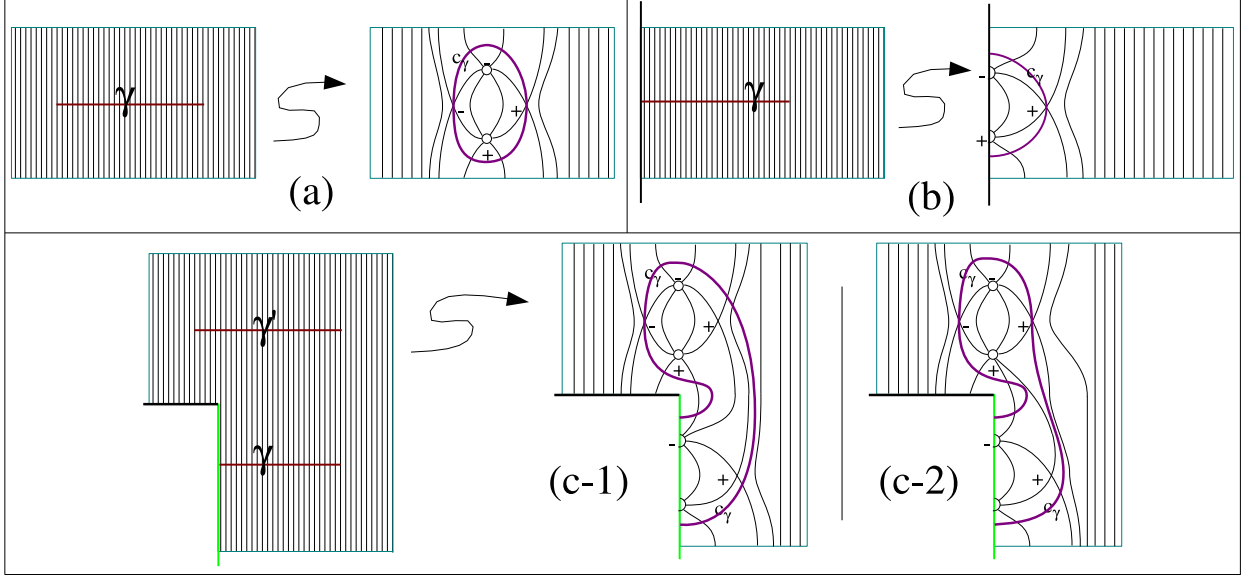


Figure 13: The parity signs indict one possibility for the parity assignment of singularities and vertices. In illustration (a) there is always the occurrence of a positive/negative pair of vertices and singularities. In illustration (b) there is always the occurrence of a positive/negative pair of vertices. The lone singularity can be of either parity. The curves $c_\gamma (\subset C_1 \cap T\Delta_\varepsilon^\eta)$ is useful in grouping of the vertices and singularities introduced by the tiling isotopy associated with the intersection arc γ .

half disc. The previously described tiling isotopy will isotopy one of the ∂ -vertical arcs of v^∂ of X_T^η . This will introduce two points of intersection of v^∂ with \mathbf{A} and one tangency point with a disc fiber of \mathbf{H} which we again call a *singular point*. Figure 12(b)-right has this corresponding alteration to the foliation of Δ_ε^η . For convenience we will assume in all situations that the singularities introduced in tiling isotopy does not occur in a disc fiber of \mathbf{H} that contains a vertical arc or ∂ -vertical arc. Again, turning to Figure 13(b) we see how the tiling isotopy alters the radial foliation in a neighborhood of such an intersection arc γ .

When half disc h is associated with is split off by a case-2 type-1 **sa** then the coning disc δ^h will intersect Δ_ε^η twice due to the “fold” in Δ_ε^η near the ∂ -vertical arc. Figure 13(c)-left illustrates the local intersecting arcs in the radial foliation of Δ_ε^η . This situation, again bifurcates into two possibilities since either γ' (the arc adjacent to a ∂ -vertical arc) is closest to \mathbf{A} or γ (the arc not adjacent to a ∂ -vertical arc) is closest to \mathbf{A} . Pushing these intersecting arcs through δ^h yields to a change in foliation that corresponds to either Figure 13(c-1) (when γ is nearest to \mathbf{A}) or Figure 13(c-2) (when γ' is nearest to \mathbf{A}).

In general, for a single choice of a coning disc δ^h we will use the notation $(X_T^\eta, \Delta_\varepsilon^\eta) \xrightarrow{T} (X_T^\eta, T\Delta_\varepsilon^\eta)$ to indicate the tiling isotopy between $(X_T^\eta, \Delta_\varepsilon^\eta)$ and $(X_T^\eta, T\Delta_\varepsilon^\eta)$. (Our arc presentation and

shearing intervals remain unchanged.) The **scc** /**sa** illustrated in Figure 12 corresponds to $c_\gamma \subset C_1 \cap T\Delta_\varepsilon^\eta$. Specifically, when c_γ is a circle it will encircle the two vertices and pass through the two singularities. When c_γ is a case-1 type-1 **sa** it will split-off the two vertices that are on the ∂ -vertical arc and will pass through the single singularity. When c_γ is a case-2 type-1 **sa** the disc it splits off will have four vertices—two on a ∂ -vertical arc and two in the interior of $T\Delta_\varepsilon^\eta$ —and three singularities. In Figure 13 we see how the tiling isotopy alters the radial foliation in a neighborhood of the intersection arc γ .

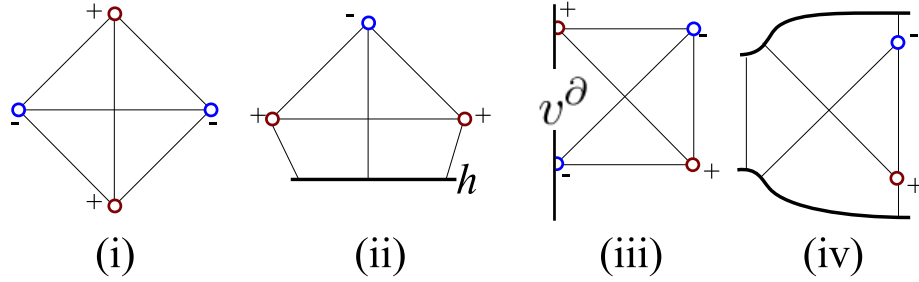


Figure 14: There are three possible means by which a singularity can be formed. In (i) two b -arcs (leaves in the foliation that have both endpoints on vertices) can come together to form a singularity. In (ii) the neighborhood of the singularity has both a -arcs (leaves having an endpoint on a vertex and an endpoint on the boundary of $T\Delta_\varepsilon^\eta$) and b -arcs. In (iii) the singularity has its vertices endpoints on a ∂ -vertical arc. In (iv) the graphs in bs -singularities are illustrated.

To review a little more thoroughly the tiling machinery, we consider the orientations of the axis \mathbf{A} , our disc $T\Delta_\varepsilon^\eta$ and the disc fibers of \mathbf{H} which is consistent with the orientation of \mathbf{A} . The orientation of $T\Delta_\varepsilon^\eta$ is consistent with the orientation of the horizontal/vertical arcs on its boundary. We can associate to each vertex and singularity a parity as follows. A vertex $\mathbf{v} \in T\Delta_\varepsilon^\eta \cap \mathbf{A}$ is *positive* (or $+$) if \mathbf{v} is a positive intersection. Otherwise, \mathbf{v} is *negative* (or $-$). A singular point $\mathbf{s} \in T\Delta_\varepsilon^\eta$ is *positive* (or $+$) if the orientation of the tangent plane to Δ_ε^η at \mathbf{s} agrees with the orientation of the disc fiber of \mathbf{H} that contains \mathbf{s} . Otherwise, \mathbf{s} is *negative* (or $-$). Thus, parity labeling of the vertices in Figure 13 are the only possible assignments, whereas the parity labeling of the singularities can either be as indicated or reversed.

Keeping with the literature that has developed around tiled foliations, on the components of $T\Delta_\varepsilon^\eta \setminus \text{int}(\mathcal{R})$ non-singular leaves must be arcs that have their endpoints being either vertices of the foliation, or points in the set $(X_T^\eta \cap C_1) \cup (\partial\mathcal{R} \cap C_1)$, i.e. horizontal arcs. Due to reasons of orientation it is easily established that there are three types of generic leaves in our tiled foliation: a -arcs which have one endpoint being a vertex and one endpoint in the set $(X_T^\eta \cap C_1) \cup (\partial\mathcal{R} \cap C_1)$; b -arcs which have both endpoints being (differing) vertices; and, s -arcs which have endpoints on differing horizontal arcs of $(X_T^\eta \cap C_1) \cup (\partial\mathcal{R} \cap C_1)$.

Again, for reasons of orientation it is easily established that there are only three types of singularities. (See Figure 14.) An ab -singularity or ab -tile is formed by an a -arc and b -arc coming together as illustrated in (ii) of Figure 14. A bb -singularity or bb -tile is formed by two b -arcs coming together as illustrated in (i) & (iii) of Figure 14. Finally, a bs -singular or bs -tile is formed by a b -arc and s -arc coming together. Notice that we could also refer to this singularity as an aa -singular or aa -tile since the two a -arcs, a b -arc and an s -arc make up the four sides of the tile. Illustration (iv) depicts this third singularity.

We note that one subdisc component of $T_\varepsilon^\eta \setminus \text{int}(\mathcal{R})$ will contain D_{+1} . All the remaining components, for which D_0 will be convenient notation, have 0 algebraic intersection with \mathbf{A} .

We now have the following tiling operations.

\mathcal{O}_1 –Simplification moves on the middle boundary. We start with a simply alteration to horizontal arcs.

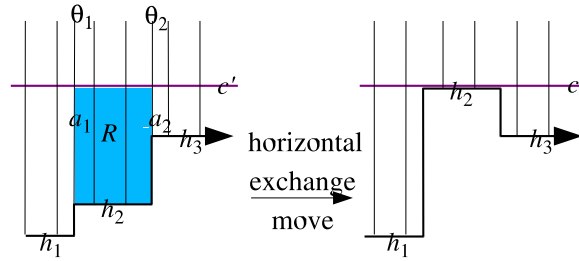


Figure 15: The sequence illustrates a horizontal exchange move that alters T_ε^η near c' .

$\mathcal{O}_{1.1}$ –Horizontal arc near a horizontal boundary arc of \mathcal{R} or c_{max} . We refer to Figure 15 for this configuration and appeal to its labeling. Let $R \subset \Delta_\varepsilon^\eta$ be a rectangular region that satisfies the following: i) $\text{int}(R) \cap C_1 = \emptyset$; ii) $\partial R = a_1 \cup c' \cup a_2 \cup h_2 \cup$ where; iii) a_1 and a_2 are subarcs of leaves containing vertical arcs, v_1 and v_2 , respectively, of X_T^η ; iv) c' is an subarc of either a horizontal boundary arc of a rectangle or c_{max} ; and, v) $h_2 \subset X_T^\eta$ is a horizontal arc. Figure 15 implies an isotopy of h_2 through R so as to place the horizontal position of the resulting horizontal arc nearest that of c' . Such an isotopy will leave all other features of $\Delta_\varepsilon^\eta \cap C_1$ unaltered. We wish to show that this isotopy can be achieved through a sequence of our elementary moves.

The horizontal position of v_1 and v_2 leads to four cases since each vertical arc can have vertical support that is either above or below the horizontal position of h_2 . The case illustrated in Figure 15 has the vertical support of v_1 below h_2 whereas v_2 has vertical support above the horizontal position of h_2 .

In Figure 15 the angle at which a_1 occurs is θ_1 and the angle at which a_2 occurs is θ_2 with $[\theta_1, \theta_2]$ being the angular support of R . Having c' being a subarc of either a horizontal boundary arc of a rectangle of \mathcal{R} or of c_{max} allows us to speak of the horizontal position of c' and the

vertical support of R (which is between the horizontal positions of c' and h_2).

Now we observe that the existence of $R \subset \Delta_\varepsilon^\eta$ forces the vertical support of every vertical arc of $X_\mathcal{I}^\eta$ having angular position in (θ_1, θ_2) to be either totally below, totally above, contained in, or properly containing the vertical support of R . (If c' is in c_{max} then there is no ‘totally above’.) Thus, for any two consecutive vertical arcs $v, v' \subset X_\mathcal{I}^\eta$ having angular position in (θ_1, θ_2) , if v has vertical support contained in the vertical support of R but v' does not then v & v' are nested and we can move v' (forward/backward) past v . Thus, the only obstruction to moving vertical arcs having angular position in (θ_1, θ_2) past θ_1 (in the backward direction) or past θ_2 (in the forward direction) are the vertical arcs v_1 and v_2 . This is where our previously mention four cases come into play. If v_1 (respectively, v_2) has vertical support below (respectively, above) h_1 (as illustrated in 15) then we can move all vertical arcs that are above (respectively, below) h_1 with angular position in (θ_1, θ_2) backwards (respectively, forward) and past v_1 (respectively, v_2). The remaining three cases are easily listed and we leave them to the reader.

The conclusions are that after some number of vertical exchange moves we can assume that over the angular interval (θ_1, θ_2) either: 1) there are not vertical arcs of $X_\mathcal{I}^\eta$; 2) all vertical arcs of $X_\mathcal{I}^\eta$ have vertical support below h_2 ; or 3) all vertical arcs of $X_\mathcal{I}^\eta$ have vertical support contained in the vertical support of R .

With these conclusions in place we now see how the isotopy depicted in Figure 15 is realized through a sequence of horizontal exchange moves. If we have conclusion 1 or 3 we can through a sequence of second flavor horizontal exchange moves move h_2 to be below and consecutive with the horizontal position of c' . If we have conclusion 3 then through a sequence of the second flavor horizontal exchange moves we can make the horizontal position of h_2 be minimal. We then perform a first flavor horizontal exchange move to make it maximal. We follow this by a sequence of second flavor horizontal exchange moves to place the horizontal position of h_2 below and consecutive with the horizontal position of c' . This may place the horizontal position of h_2 within the vertical support of a rectangle of \mathcal{R} . After h_2 is positioned near c' we perform the inverse of all of the vertical exchange moves so as to place all of the vertical arcs back in their original angular position. (Thus, the isotopy of h_2 through R is achieved by sequence of horizontal and vertical exchange moves.) The new Δ_ε^η will be such that $|\Delta_\varepsilon^\eta \cap C_1| = |\Delta_\varepsilon'^\eta \cap C_1|$. \diamond

Building on $\mathcal{O}_{1,1}$, the remaining \mathcal{O}_1 operations will allow us to simplify the middle boundary of $\partial\Delta_\varepsilon^\eta$. In particular, for the description that follows we assume that no component of $\Delta_\varepsilon^\eta \setminus (C_1 \cap \Delta_\varepsilon^\eta)$ is a whole or half disc. The middle boundary will be in a component $D^m \subset \Delta_\varepsilon^\eta \setminus \{D_{+1} \cup \mathcal{R}\}$. To list the possibilities, D^m is either a disc or an annulus. Moreover, ∂D^m intersects either one, two or three ∂ -vertical arcs of $\partial\Delta_\varepsilon^\eta$. If ∂D^m intersects one ∂ -vertical arc then D^m is an annulus and Δ_ε^η is a destabilizing disc. If D^m intersects two ∂ -vertical arcs and is an annulus then Δ_ε^η is an exchange disc. However, for two ∂ -vertical arcs if D^m is a disc then Δ_ε^η can be either an

exchange or flyping disc. If D^m intersects three ∂ -vertical arcs then Δ_ε^η is a flyping disc and D^m can be either an annulus or a disc. Using the 2-tuple subscript $\{-, t\} \in \{\{A, D\}, \{1, 2, 3\}\}$ (where A = annulus, D = disc and t is number of vertical arcs), in the above order just described we have that $D_{-,t}^m$ can be $D_{A,1}^m \subset \Delta_d^\eta$, $D_{A,2}^m \subset \Delta_e^\eta$, $D_{D,2}^m \subset \Delta_e^\eta$ or Δ_f^η , and $D_{A,3}^m$ or $D_{D,3}^m \subset \Delta_f^\eta$.

$\mathcal{O}_{1.2}$ -Simplifying $D_{A,1}^m \subset \Delta_d^\eta$. To start we observe that the **scc** 's of $\partial D_{A,1}^m$ are c_{max} and

$$\partial \Delta_d^\eta = h_1^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} \dots \dot{\cup} h_l^\eta \dot{\cup} v_l^\eta \dot{\cup}.$$

(The reader may wish to refer back to Lemma 8.) Our \mathcal{O}_1 operation assumption on the occurrence of whole and half discs gives us that $c_{max} = \text{int}(\Delta_d^\eta) \cap C_1$.

There are in fact two cases to argue since the arc $\alpha_{\theta_1}^\eta$ can be either a front edge or a back edge. For the argument below we will assume that we have a back edge. We will alert the reader to where the argument differs for a front edge but leave the details to the reader.

The steps for simplifying $\partial \Delta_d^\eta$ are as follows.

Step-1: Through a sequence of vertical and horizontal exchange moves, we reposition horizontal arcs $h_3^\eta, \dots, h_l^\eta$ such that for each their horizontal position is arbitrarily close to and below that of c_{max} .

This step is readily achieved by applying operation $\mathcal{O}_{1.1}$ to each horizontal arc in the middle boundary that is not adjacent to the ∂ -vertical arc $\alpha_{\theta_1}^\eta$. We will then have that for each horizontal arc, $h_3^\eta, \dots, h_l^\eta$, over its angular support it is consecutive with c_{max} , i.e. arbitrarily close to and below that of c_{max} .

Step-2: Through a sequence of horizontal simplifications we reduce $\partial \Delta_d^\eta$ to

$$h_1^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} h_3^\eta \dot{\cup} v_3^\eta \dot{\cup}$$

where the horizontal position of h_3^η is below and consecutive with that of c_{max} .

Having the repositioned $\partial \Delta_\varepsilon^\eta$ in hand from Step-1, we now realize that over the angular support of the edgepath $h_{l-1}^\eta \dot{\cup} v_{l-1}^\eta \dot{\cup} h_l^\eta$ we have that the horizontal positions of h_{l-1}^η and h_l^η are consecutive. We can thus perform a horizontal simplification reducing the number of vertical arcs of $X_\mathcal{I}^\eta$ by one and shortening the middle boundary. We iterate this procedure until we have $\partial \Delta_d^\eta$ to $h_1^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} h_3^\eta \dot{\cup} v_3^\eta \dot{\cup}$. By construction the horizontal position of h_3^η is below and consecutive with that of c_{max} .

Step-3: Through a sequence of vertical exchange moves followed by a shear vertical simplification we reduce $\partial \Delta_d^\eta$ to $h_3^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup}$ where the horizontal position of h_3^η is below and consecutive with that of c_{max} .

We use the Δ_ε^η coming out of Step-2 and giving us $\partial \Delta_\varepsilon^\eta = h_1^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} h_3^\eta \dot{\cup} v_3^\eta \dot{\cup}$. In Step-1 & -2 we did not use our assumption that $\alpha_{\theta_1}^\eta$ is a back edge. Here is where it comes into

play. In particular, this assumption (along with having no whole or half discs) implies that our horizontal positions for the horizontal arcs of $\partial\Delta_\varepsilon^\eta$ has ordering $z^{h_2} < z^{h_1} < z^{h_3} < z_{max}$. (A back edge assumption would result in a $z^{h_1} < z^{h_2} < z^{h_3} < z_{max}$ ordering.) From this ordering we can conclude that there exists a rectangular subdisc $R \subset \Delta_\varepsilon^\eta \cap \mathcal{T}_\infty$ having the following features.

1. $\partial R = v_3 \dot{\cup} h_1 \dot{\cup} \gamma^\partial \dot{\cup} \gamma^h \dot{\cup}$.
2. $\gamma^\partial \subset H_{\theta_1}$, the disc fiber of \mathbf{H} containing $\alpha_{\theta_1}^\eta$ and has vertical support equal to that of v_3 .
3. γ^h has horizontal position equal to that of h_3^η and angular support equal to that of h_1^η .
4. From features 2 & 3 we observe that the vertical support of R is the same as that of v_3^η .

We now use R to push v_3^η forward and into the shearing interval thus performing a shear vertical simplification. Similar to the argument given for operation $\mathcal{O}_{1.1}$ the subdisc R gives us that vertical arcs in the angular support of R will be nested with v_3^η . This isotopy which is just pushing v_3^η through the leaves of the induced foliation on R will correspond to performing a sequence of vertical exchange moves followed by a shear vertical simplification.

For a front edge assumption a similar R can be seen however it involves h_2^η and v_2^η with v_2^η being pushed backwards. \diamond

$\mathcal{O}_{1.3}$ —Simplifying $D_{A,2}^m \subset \Delta_\varepsilon^\eta$. Here, we observe that $\partial D_{A,2}^m$ is a c_{max} union

$$\partial\Delta_\varepsilon^\eta = h_1^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} \dots \dot{\cup} h_l^\eta \dot{\cup} \alpha_{\theta_2}^\eta \dot{\cup}.$$

(See Lemma 8.) Again, our \mathcal{O}_1 operation assumption gives us that $c_{max} = \text{int}(\Delta_d^\eta) \cap C_1$.

As in $\mathcal{O}_{1.2}$ there two cases: $\alpha_{\theta_1}^\eta$ is a back edge and $\alpha_{\theta_2}^\eta$ is a front edge; or the edge assignments are reversed. For the argument below we will assume that we have the former edge assignment. We will alert the reader to where the argument differs for the latter edge assignment but leave the details to the reader.

The steps for simplifying $\partial\Delta_\varepsilon^\eta$ are as follows.

Step-1: Through a sequence of vertical and horizontal exchange moves, we reposition horizontal arcs $h_3^\eta, \dots, h_{l-1}^\eta$ such that for each one we have that over its angular support its horizontal position is minimal.

The step is achieved in essentially the same manner as Step-1 of $\mathcal{O}_{1.2}$ by utilizing $\mathcal{O}_{1.1}$. However, this will repositioned all our horizontal arcs near c_{max} . (If we were arguing the latter edge assignment, we would be interested in achieving this repositioning and would be done.) To achieve minimal horizontal position over the angular support of one of our horizontal arcs

we perform a first flavor horizontal exchange move. After a sequence of such first flavor moves we achieve the goal of this step.

If it happens that our initial middle boundary has exactly two horizontal arcs then we skip this step and Step-2, and proceed to Step-3.

Step-2: Through a sequence of horizontal simplifications we reduce $\partial\Delta_e^\eta$ to

$$h_1^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} h_3^\eta \dot{\cup} v_3^\eta \dot{\cup} h_4^\eta \dot{\cup} \alpha_{\theta_2}^\eta \dot{\cup}$$

where h_3^η has minimal horizontal position.

This step is a repeat of the argument of Step-2 of $\mathcal{O}_{1.2}$ except that all of our horizontal arcs are at a minimal (not maximal) horizontal position in their angular support.

For the latter edge assignment case we would end up with h_3^η being at maximal horizontal position and below c_{max} .

Step-3: Through a sequence of vertical exchange moves and shear vertical simplifications we reduce $\partial\Delta_e^\eta$ to $h_1^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} \alpha_{\theta_2}^\eta \dot{\cup}$.

We again adapt operation $\mathcal{O}_{1.1}$ to the resulting Step-2 middle boundary. Specifically, we consider the edgepath $\alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta$. Observe that the vertical support of $\alpha_{\theta_1}^\eta$ is above the horizontal position of h_2^η , whereas the vertical support of v_2^η is below the position of h_2^η . So we can push any vertical arc having vertical support below h_2^η and angular position in the angular support of h_2^η backwards and past $\alpha_{\theta_1}^\eta$. But, pushing them past $\alpha_{\theta_1}^\eta$ is pushing them into a shearing interval. So through a sequence of vertical exchange moves and shear vertical simplification we can alter X_T^η so as to assume that there are no vertical arcs below the horizontal position of h_2^η in the angular support of h_2^η .

Once this is achieved we consider the *virtual rectangular disc* $R = [\theta_1^{h_2^\eta}, \theta_2^{h_2^\eta}] \times v_2^\eta \subset \mathbb{R}^3$, where the angular interval is the angular support of h_2^η . Since we have no vertical arcs of X_T^η below and in the angular support of h_2^η we have that $\text{int}(R) \cap \Delta_e^\eta = \emptyset$. We can then utilize our virtual R in a similar manner to our R -subdisc in Step-3 of $\mathcal{O}_{1.2}$ to push v_2^η backwards into the shearing interval associated with $\alpha_{\theta_1}^\eta$. Arguing in a similar fashion but with a forward push of v_3^η past $\alpha_{\theta_2}^\eta$ we can achieve the $\partial\Delta_e^\eta$ described in this step.

If our initial middle boundary had only two horizontal arcs then we can still construct a virtual rectangular disc and we will push the single vertical arc in the direction of the shortest (in the vertical support sense) ∂ -vertical arc.

For the latter edge assignment case we can use R -subdiscs similar to the one in Step-3 of $\mathcal{O}_{1.2}$. ◇

$\mathcal{O}_{1.4}$ —Simplifying $D_{D,2}^m \subset \Delta_e^\eta$ or Δ_f^η . Here we have $\partial D_{D,2}^m$ being of the form

$$\gamma_+ \dot{\cup} \gamma_1^\partial \dot{\cup} h_2^\eta \dot{\cup} v_1^\eta \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} \dots \dot{\cup} h_l^\eta \dot{\cup} \gamma_2^\partial \dot{\cup}$$

were γ_+ is a horizontal boundary of a rectangle of \mathcal{R} , and γ_1^∂ & γ_2^∂ are subarcs in distinct ∂ -vertical arcs of Δ_e^η . Again, our \mathcal{O}_1 operation assumption gives us that $\text{int}(D_{D,2}^m) \cap C_1 = \emptyset$. The steps for simplifying $\partial \Delta_e^\eta$ are as follows.

By our assumptions in our discussion after Lemma 8 on the start/end of the middle boundary we know that for Δ_e^η we must have γ_+ being a type-2 **sa** and a type-3 **sa** for Δ_f^η . For the Δ_f^η case there are two cases since we have either a positive or negative flype. We will assume a positive flype and leave the similar negative case to the reader.

For the case of $D_{D,2}^m \subset \Delta_e^\eta$ we appeal to our $\mathcal{O}_{1.3}$ argument as a model. As just observed, we have that γ_1^∂ is a subarc on a back edge and γ_2^∂ is on a front edge. Then imitating Step-1 of $\mathcal{O}_{1.3}$ we can use operation $\mathcal{O}_{1.1}$ to reposition all of the horizontal edges of $D_{D,2}^m \cap X_{\mathcal{I}}^\eta$ to be near γ_+ . Continuing and imitating Step-2 of $\mathcal{O}_{1.3}$ we can perform horizontal simplifications to alter $\partial D_{D,2}^m$ to be $\gamma_+ \dot{\cup} \gamma_1^\partial \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} h_3^\eta \dot{\cup} v_3^\eta \dot{\cup} h_4^\eta \dot{\cup} v_2^\partial \dot{\cup}$. Finally, we can imitate Step-3 of $\mathcal{O}_{1.3}$ altering $\partial D_{D,2}^m$ to be $\gamma_+ \dot{\cup} \gamma_1^\partial \dot{\cup} h_2^\eta \dot{\cup} \gamma_2^\partial \dot{\cup}$ through a sequence of vertical exchange moves and vertical simplifications.

We also appeal to $\mathcal{O}_{1.3}$ in the $D_{D,2}^m \subset \Delta_f^\eta$ positive flype case with γ_1^∂ & γ_2^∂ both being subarcs on differing front edges. Again, imitating Step-1 of $\mathcal{O}_{1.3}$ we can use operation $\mathcal{O}_{1.1}$ to reposition all of the horizontal edges of $D_{D,2}^m \cap X_{\mathcal{I}}^\eta$ to be near γ_+ . Continuing and imitating Step-2 of $\mathcal{O}_{1.3}$ we can perform horizontal simplifications to alter $\partial D_{D,2}^m$ to be $\gamma_+ \dot{\cup} \gamma_1^\partial \dot{\cup} h_2^\eta \dot{\cup} v_2^\eta \dot{\cup} h_3^\eta \dot{\cup} v_3^\eta \dot{\cup} h_4^\eta \dot{\cup} v_2^\partial \dot{\cup}$. Finally, we can imitate Step-3 of $\mathcal{O}_{1.3}$ altering $\partial D_{D,2}^m$ to be $\gamma_+ \dot{\cup} \gamma_1^\partial \dot{\cup} h_2^\eta \dot{\cup} \gamma_2^\partial \dot{\cup}$ through a sequence of vertical exchange moves and vertical simplifications. However, this totally simplifies the middle boundary to one horizontal arc. Thus, the resulting notch disc cannot imply the existence of a flype. So we could not have had D_0 be contained in a flying disc. \diamond

$\mathcal{O}_{1.5}$ —Simplifying $D_{A,3}^m \subset \Delta_f^\eta$. Briefly, by Lemma 8 statement d. we have $\partial D_{D,3}^m$ is the union of c_{max} and

$$h_1^\eta \dot{\cup} \alpha_{\theta_1}^\eta \dot{\cup} h_2^\eta \dot{\cup} \alpha_{\theta_2}^\eta \dot{\cup} h_3^\eta \dot{\cup} v_3^\eta \dot{\cup} \dots \dot{\cup} h_l^\eta \dot{\cup} \alpha_{\theta_3}^\eta \dot{\cup}$$

Again, the \mathcal{O}_1 operation assumption gives us that $c_{max} = \text{int}(\Delta_d^\eta) \cap C_1$.

Our edge assignment assumption for Δ_f^η (see FA-a) gives us two similar but different cases to argue. Either $\alpha_{\theta_1}^\eta$, $\alpha_{\theta_2}^\eta$ and $\alpha_{\theta_3}^\eta$ are a front edge, back edge, front edge, respectively; or, they are a back edge, front edge and back edge, respectively. We will argument the former case (the positive flype case) and, with hints, leave the latter case to the reader.

The steps for simplifying $\partial \Delta_d^\eta$ imitates the steps for $\mathcal{O}_{1.2}$. Specifically, starting with the middle boundary $h_3^\eta \dot{\cup} v_3^\eta \dot{\cup} \dots \dot{\cup} h_l^\eta$ by a sequence vertical and horizontal exchange moves we can reposition horizontal arcs h_4^η through h_{l-1}^η so that each has minimal horizontal position over its

angular support. Next, by a sequence of horizontal simplifications we can shorten the middle boundary until it corresponds to $h_3^\eta \dot{\cup} v_3^\eta \dot{\cup} h_4^\eta \dot{\cup} v_4^\eta \dot{\cup} h_5^\eta$ with the horizontal position of h_4^η being minimal. Finally, using our back edge assumption for $\alpha_{\theta_2}^\eta$ and $\alpha_{\theta_3}^\eta$ we realize that we can imitate the argument in Step-3 of $\mathcal{O}_{1.3}$ to construct a virtual rectangular disc for pushing v_3^η forward and past $\alpha_{\theta_1}^\eta$. As in Step-3 of $\mathcal{O}_{1.3}$, this will be achieved through a sequence of vertical exchange moves and shear vertical simplifications. We then obtain $h_3^\eta \dot{\cup} v_3^\eta \dot{\cup} h_4^\eta$ as the middle boundary.

In the front-front edge assignment case we would construct a virtual rectangular disc for pushing v_4^η backward past $\alpha_{\theta_3}^\eta$. \diamond

$\mathcal{O}_{1.6}$ –Simplifying $D_{D,3}^m \subset \Delta_f^\eta$. This remaining simplification is handled exactly as the previous simplification in $\mathcal{O}_{1.5}$. In middle boundary of $\partial\Delta_f^\eta$ contained in $\partial D_{D,3}^m$ becomes $h_3^\eta \dot{\cup} v_3^\eta \dot{\cup} h_4^\eta$ through a sequence of our moves from §1.2. It is useful to observe that γ_+ must be either a type-2 or -4 **sa**. \diamond

\mathcal{O}_2 –Elimination of whole discs. There are three differing configurations of the tiled foliation which can be used to detect the occurrence of horizontal and vertical exchange moves for the elimination of whole discs.

$\mathcal{O}_{2.1}$ –Elimination of a whole disc near c_{max} . We consider a coning disc δ^h having an intersection arc $\gamma \subset \delta^h \cap \Delta_\varepsilon^\eta$ which is *innermost* on the component of $\Delta_\varepsilon^\eta \setminus \mathcal{R}$ containing D_{+1} . That is, for every leaf a_θ in the radial foliation of Δ_ε^η intersecting γ we have that when traversing a_θ with \mathbf{v}_{max} as the starting point, the first time a_θ intersects any coning disc is at the point $a_\theta \cap \gamma$. For such a choice of coning disc we apply the transition $(X_\mathcal{I}^\eta, \Delta_\varepsilon^\eta) \xrightarrow{T} (X_\mathcal{I}^\eta, \Delta_\varepsilon^\eta)$ to produce our tiled notch disc. Referring back to Figure 13(a), if γ is innermost then all of the “top of the page” leaves adjacent to the negative vertex are “coned” to \mathbf{v}_{max} . Moreover, the curve c_γ will be “nearest” to c_{max} . Figure 16(a) illustrates these features. The illustrated foliated subdisc of Δ_ε^η contains \mathbf{v}_{max} with two additional vertices, \mathbf{v}_1 and \mathbf{v}_2 , and the two singularities, \mathbf{s}_1 and \mathbf{s}_2 . \mathbf{v}_{max} and \mathbf{v}_2 are positive vertices, whereas \mathbf{v}_1 has negative parity. The **scc** $c_\gamma \subset \Delta_\varepsilon^\eta \cap C_1$ passes through the two singularities and encircles \mathbf{v}_1 and \mathbf{v}_2 . Between \mathbf{v}_1 and \mathbf{v}_2 on the axis \mathbf{A} there is a vertex, \mathbf{v}^h , of coning disc δ^h for our horizontal arc $h \subset X_\mathcal{I}^\eta$. And, the b -arcs adjacent to \mathbf{v}_{max} & \mathbf{v}_1 intersect $\Delta_\varepsilon^\eta \cap C_1$ only at c_{max} and c_γ .

We now perform the classical exchange move of [BM1, BM2, BM3, BM4, BF] that isotop δ^h so that the resulting \mathbf{v}^h and \mathbf{v}_{max} are at consecutive horizontal positions. After the isotopy $|\delta^h \cap \text{int}(\Delta_\varepsilon^\eta)|$ will decrease by at least one: γ will be eliminated; and, all the arcs of $\delta^h \cap \text{int}(\Delta_\varepsilon^\eta)$ in δ^h between \mathbf{v}^h and γ will also be eliminated. The **scc**’s associated with these arcs will also be eliminated. However, any arcs of $\delta^h \cap \text{int}(\Delta_\varepsilon^\eta)$ in δ^h between γ and h will remain along with their associated **scc**. Thus, the intersection curve c_γ is eliminated to reduce $|\text{int}(\Delta_\varepsilon^\eta) \cap C_1|$, and we replace Δ_ε^η with a radially foliated notch disc Δ_ε^η .

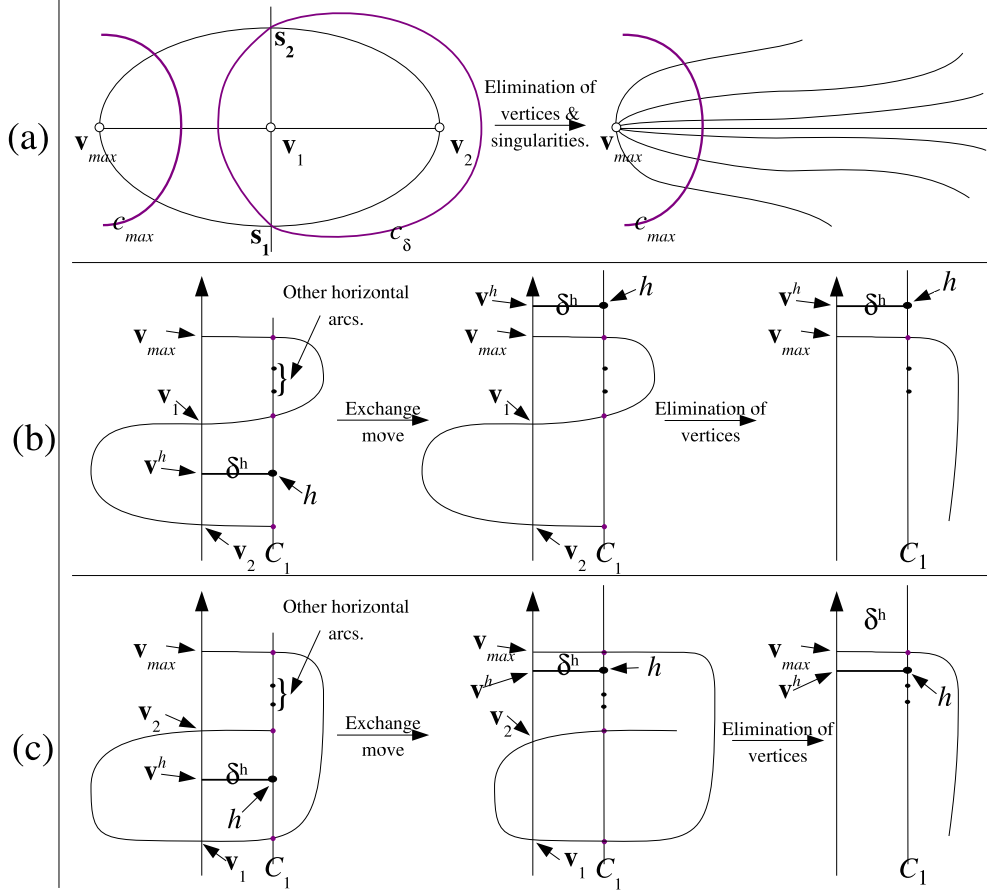


Figure 16: In (a) we depict the tiled foliation in a neighborhood of a valence two vertex and show the change to the foliation when we eliminate positive/negative pairs of vertices and singularities. In (b) we illustrate a cross-section of the neighborhood in (a) along with the cross-section of the coning disc δ^h and the cylinder C_1 in the case where the order of vertices on \mathbf{A} is $\mathbf{v}_2 < \mathbf{v}^h < \mathbf{v}_1 < \mathbf{v}_{max}$. In (c) we have a similar cross-section for the order $\mathbf{v}_1 < \mathbf{v}^h < \mathbf{v}_2 < \mathbf{v}_{max}$.

We have two cases depending on the initial order of our four vertices on \mathbf{A} : when the initial order is $\mathbf{v}_2 < \mathbf{v}^h < \mathbf{v}_1 < \mathbf{v}_{max}$ the classical exchange move will isotop δ^h so the $\mathbf{v}_{max} < \mathbf{v}^h$ (Figure 16(b)-middle); and, when the initial order is $\mathbf{v}_1 < \mathbf{v}^h < \mathbf{v}_2 < \mathbf{v}_{max}$ the classical exchange move will isotop δ^h so the $\mathbf{v}^h < \mathbf{v}_{max}$ (Figure 16(c)-middle). (Treating vertices as their horizontal-position's real values on \mathbf{A} is a convenient abuse of notation.) In the Figure 16(b) case s_2 is necessarily a negative singularity and s_1 is positive. Whereas in the Figure 16(c) case s_2 is positive and s_1 is negative. Our difficulty is in seeing that these two isotopy cases in fact correspond to a sequence of horizontal and vertical exchange moves as defined in §1.2. In particular, for the second flavor of horizontal exchange moves the key features of consecutive and nested have to be verified. This verification will be similar but different for the

Figure 16(b) and (c) cases.

For convenience, let \mathbf{s}_1 and \mathbf{s}_2 have angular position θ_1 and θ_2 respectively. Then the b -arcs adjacent to \mathbf{v}_{max} and \mathbf{v}_1 occur over the angular interval $[\theta_1, \theta_2] \subset S^1$. Moreover, the angular support of h , $[\theta_1^h, \theta_2^h]$, is properly contained in but arbitrarily close to $[\theta_1, \theta_2]$.

We start with the case in Figure 16(b). If each horizontal arc of $X_{\mathcal{I}}^\eta$ having horizontal position in the interval $(\mathbf{v}_1, \mathbf{v}_{max}) \subset \mathbf{A}$ also has angular support containing the interval $[\theta_1^h, \theta_2^h]$ then the nested feature is satisfied and the isotopy in (b) is just a sequence of horizontal exchange moves (second flavor).

Now, suppose that $h' \subset X_{\mathcal{I}}^\eta$ is a horizontal arc having horizontal position in the interval $(\mathbf{v}_1, \mathbf{v}_{max})$ and angular support $[\theta_1^{h'}, \theta_2^{h'}]$ with $\theta_2^{h'} \in (\theta_1^h, \theta_2^h)$. Without loss of generality we can assume that the angle $\theta_2^{h'}$ is the nearest such angle to θ_2^h , and let $v' \subset X_{\mathcal{I}}^\eta \cap H_{\theta_2^{h'}}$ be the vertical arc adjacent to h' . Since the b -arcs adjacent to \mathbf{v}_1 and \mathbf{v}_{max} have vertical support $[\mathbf{v}_1, \mathbf{v}_{max}] \subset \mathbf{A}$, the vertical support of v' must be contained in $(\mathbf{v}_1, \mathbf{v}_{max})$, i.e. $v' \cap \text{int}(\tilde{\Delta}_\varepsilon^\eta) = \emptyset$. Moreover, by our ' $\theta_2^{h'}$ nearest θ_2^h ' assumption, all other vertical arcs of $X_{\mathcal{I}}^\eta$ having angular position inside the interval $(\theta_2^{h'}, \theta_2^h)$ must have vertical support strictly below \mathbf{v}_1 . Thus, the key feature of nested for vertical exchange moves is satisfied for v' and the next consecutive vertical arc in the angular interval $(\theta_2^{h'}, \theta_2^h)$. Moving v' past this arc and iterating this procedure, through a sequence of vertical exchange moves we can push v' forward and outside the interval $[\theta_1^h, \theta_2^h]$. Thus, through a sequence of vertical exchange moves we can assume that the angular support of all horizontal arcs having horizontal position in $(\mathbf{v}_1, \mathbf{v}_{max})$ properly contains $[\theta_1^h, \theta_2^h]$. The isotopy of Figure 16(a) can thus be achieved through a sequence of vertical and horizontal exchange moves. Moreover, once we have placed h above that of \mathbf{v}_{max} we perform the inverse of all of the vertical exchange moves to place all of the vertical arcs back in their original angular position.

But, we are not quite done since the isotopy in Figure 16(a) leaves the horizontal position of h above that of \mathbf{v}_{max} . To resolve this remaining issue we note that the horizontal position of h is now maximal and we can perform a first flavor horizontal exchange move on h making its horizontal position minimal. Notice this will not change the value of $|\Delta_\varepsilon^\eta \cap C_1|$. Moreover, since a first flavor horizontal exchange move on h corresponds to re-choosing the point at infinity on \mathbf{A} (a choice between the horizontal positions of \mathbf{v}_{max} and h), the foliation of Δ_ε^η and its new intersection with C_1 remains the same after this exchange move.

We next deal with the case in Figure 16(c). It is argued in a similar fashion to the previous case except we work with arcs of $X_{\mathcal{I}}^\eta$ whose horizontal position and vertical support are below that of \mathbf{v}_1 . Specifically, if each horizontal arc of $X_{\mathcal{I}}^\eta$ having horizontal position in the interval $(-\infty, \mathbf{v}_1) \subset \mathbf{A}$ also has angular support containing the interval $[\theta_1^h, \theta_2^h]$ then the nested feature is satisfied and the isotopy in (b) is just a sequence of second flavor horizontal exchange moves

followed by a single first flavor horizontal exchange move.

Similar to before, suppose that $h' \subset X_{\mathcal{I}}^{\eta}$ is a horizontal arc having horizontal position in the interval $(-\infty, \mathbf{v}_1)$ and angular support $[\theta_1^{h'}, \theta_2^{h'}]$ with $\theta_2^{h'} \in (\theta_1^h, \theta_2^h)$. Again, without loss of generality we can assume that the angle $\theta_2^{h'}$ is the nearest such angle to θ_2^h , and let $v' \subset X_{\mathcal{I}}^{\eta} \cap H_{\theta_2^{h'}}$ be the vertical arc adjacent to h' . Since the b -arcs adjacent to \mathbf{v}_1 and \mathbf{v}_{max} have vertical support $[\mathbf{v}_1, \mathbf{v}_{max}] \subset \mathbf{A}$, the vertical support of v' must be contained in $(-\infty, \mathbf{v}_1)$, i.e. $v' \cap \text{int}(\Delta_{\varepsilon}^{\eta}) = \emptyset$. As before, all other vertical arcs of $X_{\mathcal{I}}^{\eta}$ having angular position inside the interval $(\theta_2^{h'}, \theta_2^h)$ must have vertical support strictly above \mathbf{v}_1 . Thus, the key feature of nested for vertical exchange moves is satisfied for v' and the next consecutive vertical arc in the angular interval $(\theta_2^{h'}, \theta_2^h)$. Moving v' past this arc and iterating this procedure, through a sequence of vertical exchange moves we can push v' forward and outside the interval $[\theta_1^h, \theta_2^h]$. Thus, through a sequence of vertical exchange moves we can assume that the angular support of all horizontal arcs having horizontal position in $(-\infty, \mathbf{v}_1)$ properly contains $[\theta_1^h, \theta_2^h]$. Once we have placed h below that of \mathbf{v}_{max} , we perform the inverse of all of the vertical exchange moves to place all of the vertical arcs back in their original angular position. The isotopy of Figure 16(a) can be achieved through a sequence of vertical and horizontal exchange moves. \diamond

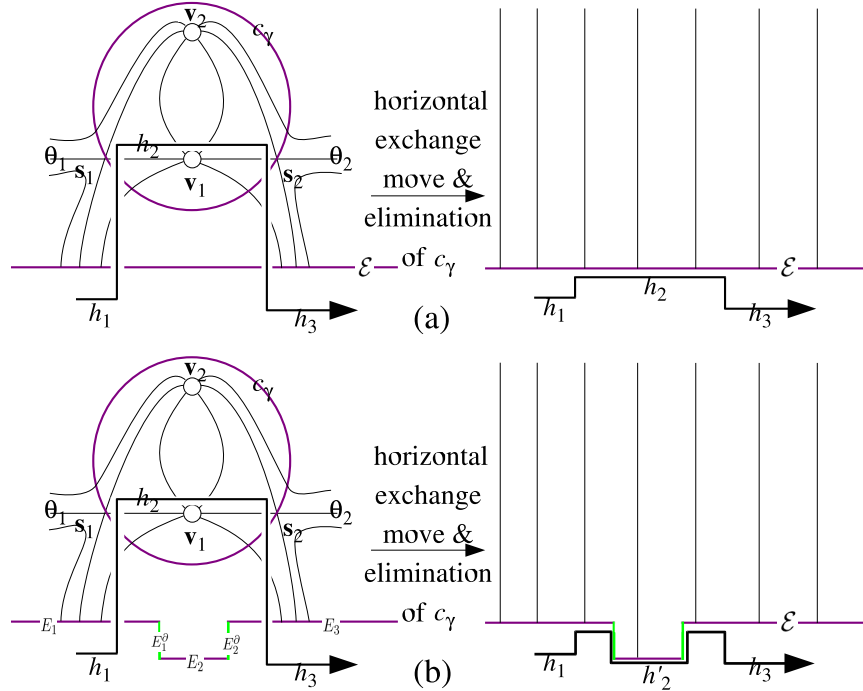


Figure 17: The top sequence illustrates the use of a horizontal exchange move in the elimination of a valence two vertex and a c_{γ} curve that is nearest \mathcal{E} . The bottom sequence illustrates shear horizontal exchange moves when cE contains ∂ -vertical arcs (possibility-B).

$\mathcal{O}_{2.2}$ —Elimination of a whole disc near an edgepath \mathcal{E} that is away from ∂ -vertical arcs. We consider a coning disc δ^h having an intersection arc $\gamma \subset \delta^h \cap \Delta_\varepsilon^\eta$ which is *nearest* on a D_0 or the D_{+1} component, $D^\star \subset \Delta_\varepsilon^\eta \setminus \text{int}(\mathcal{R})$, to an edge-path $\mathcal{E} \subset \partial D^\star$. That is, for every leaf a_θ in the radial foliation of Δ_ε^η intersecting γ we have that it also intersects \mathcal{E} and, when traversing a_θ from $a_\theta \cap \mathcal{E}$ to $a_\theta \cap \gamma$, the first time a_θ intersects any coning disc is at $a_\theta \cap \gamma$. (Picture \mathcal{E} having slightly larger angular support than that of $c_\gamma \subset \Delta_\varepsilon^\eta \cap C_1$, the **scc** associated with γ .) Finally, assume that no leaf a_θ in the radial foliation of Δ_ε^η intersecting γ also contains a ∂ -vertical arc of Δ_ε^η .

Such an edgepath \mathcal{E} is a portion of either $X_\mathcal{I}^\eta \cap D^\star$ or a horizontal boundary arc of a component of \mathcal{R} . Taking the most straight forward situations first, assume that \mathcal{E} is a portion of either a horizontal arc of $X_\mathcal{I}^\eta$ or a horizontal boundary arc of a component of \mathcal{R} .

For such a choice of coning disc we apply the transition $(X_\mathcal{I}^\eta, \Delta_\varepsilon^\eta) \xrightarrow{T} (X_\mathcal{I}^\eta, \overset{T}{\Delta}_\varepsilon^\eta)$ to produce our tiled notch disc. This choice of coning disc yields a configuration similar to that in $\mathcal{O}_{2.1}$. Referring back to Figure 13(a)-left, if γ is nearest \mathcal{E} then the curve c_γ will be “nearest” to $\mathcal{E} \subset \partial D^\star$. Figure 17(a) illustrates these features and we now appeal to this figure’s labeling. The illustrated foliated subdisc of $\overset{T}{\Delta}_\varepsilon^\eta$ has horizontal boundary arc \mathcal{E} along with two additional vertices, \mathbf{v}_1 and \mathbf{v}_2 , and the two singularities, \mathbf{s}_1 and \mathbf{s}_2 in $\text{int}(\overset{T}{\Delta}_\varepsilon^\eta)$. \mathbf{v}_2 and \mathbf{v}_1 have opposite parity. (When \mathcal{E} is a horizontal arc of $X_\mathcal{I}^\eta$ then \mathbf{v}_1 is positive and \mathbf{v}_2 is negative.) The **scc** $c_\gamma \subset \overset{T}{\Delta}_\varepsilon^\eta \cap C_1$ passes through the two singularities and encircles \mathbf{v}_1 and \mathbf{v}_2 . Between \mathbf{v}_1 and \mathbf{v}_2 on the axis \mathbf{A} there is a vertex, \mathbf{v}^{h_2} , of coning disc δ^{h_2} for our horizontal arc $h_2 \subset X_\mathcal{I}^\eta$. And, the a -arcs in the induced foliation of D^\star adjacent to \mathbf{v}_1 and having endpoints on \mathcal{E} intersect $\overset{T}{\Delta}_\varepsilon^\eta \cap C_1$ only at \mathcal{E} & c_γ .

We perform the classical boundary exchange move of [BM1, BM2, BM3, BM4, BF] that is predicated on the existence of a valence two vertex near the boundary of a surface. Figure 15(a) illustrates the isotopy and elimination of the **scc** c_γ . Specifically, we isotop δ^{h_2} so that the resulting \mathbf{v}^{h_2} and \mathcal{E} are at consecutive horizontal positions—we allow for the resulting horizontal position of \mathbf{v}^{h_2} to possibly be in the vertical support of a rectangle of \mathcal{R} when \mathcal{E} is a horizontal boundary arc of a component of \mathcal{R} . After the isotopy $|\delta^{h_2} \cap (\text{int}(\overset{T}{\Delta}_\varepsilon^\eta) \setminus \mathcal{R})|$ has been reduced by at least one: the arc γ has been eliminated; and, any arc of $\delta^{h_2} \cap (\text{int}(\overset{T}{\Delta}_\varepsilon^\eta) \setminus \mathcal{R})$ between \mathbf{v}^{h_2} and γ has been eliminated. Thus, the intersection curve c_γ is eliminated reducing $|\text{int}(\overset{T}{\Delta}_\varepsilon^\eta) \cap C_1|$, and we replace $\overset{T}{\Delta}_\varepsilon^\eta$ with a radially foliated notch disc Δ_ε^η . (Again, any **scc** of $\overset{T}{\Delta}_\varepsilon^\eta \cap C_1$ associated with the previously mentioned arcs between \mathbf{v}^{h_2} and γ is eliminated by this classical exchange move.)

The details of this argument are exactly like those for $\mathcal{O}_{2.1}$ when we substitute the horizontal position of \mathcal{E} for the horizontal position of \mathbf{v}_{max} . In fact, the cross-sections illustration in Figure 16(b) & (c) can be appealed to if we alter our understanding of the illustration by thinking of the point $\mathbf{v}_{max} \subset \mathbf{A}$ as being the cone point for a coning disc of \mathcal{E} . The isotopies of Figure

16(b) & (c) may possibly result in the horizontal position of h_2 being in the vertical support of a rectangle of \mathcal{R} . As such we will not be repetitive by imitating the argument of $\mathcal{O}_{2.1}$, but conclude that through a sequence of vertical and horizontal (both flavors) exchange moves we can eliminate the c_γ intersect curve and replace Δ_ε^η with a new Δ_ε^η having $|\Delta_\varepsilon^\eta \cap C_1| > |\Delta_\varepsilon^\eta \cap C_1|$. (We do not forget to perform the inverse of all of the vertical exchange moves so as to place all of the vertical arcs back in their original angular position.)

The remaining case is when \mathcal{E} is in X_T^η but is not a portion of a single horizontal arc. However, we can transform \mathcal{E} into a portion of such a horizontal arc by utilizing $\mathcal{O}_{1.1}$ and imitating the argument in Step-1 & Step-2 of $\mathcal{O}_{1.2}$. Such a transformation of \mathcal{E} will necessarily employ the use of horizontal and vertical exchange moves, and horizontal simplifications. \diamond

$\mathcal{O}_{2.3}$ —Elimination of a whole disc near an edgpath \mathcal{E} containing ∂ -vertical arcs. Again, we consider a coning disc δ^h having an intersection arc $\gamma \subset \delta^h \cap \Delta_\varepsilon^\eta$ which is *nearest* on a D_0 or the D_{+1} component, $D^* \subset \Delta_\varepsilon^\eta \setminus \text{int}(\mathcal{R})$, to an edge-path $\mathcal{E} \subset \partial D^*$. As before this means that for every leaf a_θ in the radial foliation of Δ_ε^η intersecting γ we have that it also intersects \mathcal{E} and, when traversing a_θ from $a_\theta \cap \mathcal{E}$ to $a_\theta \cap \gamma$, the first time a_θ intersects any coning disc is at $a_\theta \cap \gamma$. However, we now assume that there are leaves a_θ in the radial foliation of Δ_ε^η intersecting γ containing ∂ -vertical arcs of Δ_ε^η .

Over the angular support of γ we list the initial possibilities: (i) Δ_d^η with one such leaf containing a ∂ -vertical arc; (ii) Δ_e^η with one or two such leaves containing ∂ -vertical arcs; and, (iii) Δ_f^η with one, two or three such leaves containing ∂ -vertical arcs. From this list of six initial possibilities it is useful to eliminate outright three of them. In particular, for possibilities (i), (ii) with two leaves, and (iii) with three leaves, the **scc** c_γ will be *close* to c_{max} . That is, when traversing a leaf a_θ in the radial foliation of Δ_ε^η from $a_\theta \cap c_{max}$ to $a_\theta \cap c_\gamma$, a_θ will only intersect other **scc** 's of $\Delta_\varepsilon^\eta \cap C_1$, i.e. it does not intersect any **sa** . Clearly **scc** 's that are close to c_{max} can be eliminated through a sequence of $\mathcal{O}_{2.1}$ starting with the innermost first. Thus, we have to eliminate only (ii) with one leaf (call this possibility-A) and (iii) with two leaves (call this possibility-B) containing ∂ -vertical arcs.

To continue, we consider the simplest situations first. For possibility-A, let $\mathcal{E} = E_1 \dot{\cup} E^\partial \dot{\cup} E_2$ where E^∂ is a portion of a ∂ -vertical arc and E_1 & E_2 are portions of horizontal arcs of either X_T^η or $\partial\mathcal{R}$. For possibility-B, let $\mathcal{E} = E_1 \dot{\cup} E_1^\partial \dot{\cup} E_2 \dot{\cup} E_2^\partial \dot{\cup} E_3$ where E_1^∂ & E_2^∂ is a portions of a ∂ -vertical arcs and E_1 , E_2 & E_3 are portions of horizontal arcs of either X_T^η or $\partial\mathcal{R}$. (Figure 17(b)-left illustrates this simple situation for possibility-B.)

Now, we return to our choice of coning disc and we apply the transition $(X_T^\eta, \Delta_\varepsilon^\eta) \xrightarrow{T} (X_T^\eta, \overset{T}{\Delta}_\varepsilon^\eta)$ to produce our tiled notch disc. Referring back to Figure 13(b)-left, if γ is nearest \mathcal{E} then the curve c_γ will be “nearest” to $\mathcal{E} \subset \partial D^*$. Figure 17(b) illustrates these features and we now appeal to this figure’s labeling. The illustrated foliated subdisc of $\overset{T}{\Delta}_\varepsilon^\eta$ has edgpath $\mathcal{E}(=$

$E_1 \dot{\cup} E_1^\partial \dot{\cup} E_2 \dot{\cup} E_2^\partial \dot{\cup} E_3$) for the simple-possibility-B along with two additional vertices, \mathbf{v}_1 and \mathbf{v}_2 , and the two singularities, \mathbf{s}_1 and \mathbf{s}_2 in $\text{int}(\overline{\Delta}_\varepsilon^\eta)$. \mathbf{v}_2 and \mathbf{v}_1 have opposite parity. (When \mathcal{E} contains horizontal arcs of $X_\mathcal{I}^\eta$ then \mathbf{v}_1 is positive and \mathbf{v}_2 is negative.) The **scc** $c_\gamma \subset \overline{\Delta}_\varepsilon^\eta \cap C_1$ passes through the two singularities and encircles \mathbf{v}_1 and \mathbf{v}_2 . Between \mathbf{v}_1 and \mathbf{v}_2 on the axis \mathbf{A} there is a vertex, \mathbf{v}^{h_2} , of coning disc δ^{h_2} for our horizontal arc $h_2 \subset X_\mathcal{I}^\eta$. And, the a -arcs in the induced foliation of D^\star adjacent to \mathbf{v}_1 and having endpoints on \mathcal{E} intersect $\overline{\Delta}_\varepsilon^\eta \cap C_1$ only at \mathcal{E} & c_γ .

We cannot quite perform the classical boundary exchange that is predicated on the existence of a valence two vertex near the boundary of a surface since the ∂ -vertical arcs obstruct such a move. However, for each portion of a ∂ -vertical arc in our edgepath \mathcal{E} we can perform a shear horizontal exchange move. Figure 15(b) illustrates the isotopy and elimination of the **scc** c_γ . This move can be thought of as “shearing” the coning discs into two (for possibility-A) or three (for possibility-B) coning discs and then isotoping them so that the resulting sheared-coning discs and horizontal sub-arcs of \mathcal{E} are at consecutive horizontal positions. (See Figure 15(b)-right.) After the isotopy c_γ has been eliminated and for the resulting Δ_ε^η we have that $|\Delta_\varepsilon^\eta \setminus \mathcal{R}] \cap [C_1 \setminus (C_1 \cap \mathcal{I})|$ has been reduced.

Once we have sheared δ^h the details establishing that this entire operation eliminating c_γ corresponds to a sequence of our elementary moves are exactly like the details in $\mathcal{O}_{2.2}$ except that we are applying them to the newly formed horizontal arcs coming from shearing h . The remaining case is when the E_i portions of \mathcal{E} are in $X_\mathcal{I}^\eta$ but is not a portion of a single horizontal arc. However, we can transform each such E_i into a portion of such a horizontal arc by utilizing $\mathcal{O}_{1.1}$ and imitating the argument in Step-1 & Step-2 of $\mathcal{O}_{1.2}$. Such a transformation of \mathcal{E} will necessarily employ the use of horizontal and vertical exchange moves, and horizontal simplifications. \diamond

\mathcal{O}_3 –Eliminating half discs. Similar to the count of \mathcal{O}_2 operations, there are three differing configurations of the tiled foliation which can be used to detect the occurrence of our elementary moves for the elimination of half discs. Recalling Figure 11 and the arguments of Lemmas 10 and 11, we note that we have been careful to deal with both case-1 and case-2 type-1 **sa**’s individually. However, our \mathcal{O}_3 moves for eliminating half discs treat the **sa** of these two cases essentially the same. So our discussion (and accompanying illustrations) will describe the properties of these moves mostly in terms of case-1 type-1 **sa**. More important to this discussion is whether x “can see the half disc” where $x \subset \Delta_\varepsilon^\eta \cap C_1$. (The possibilities for x are c_{max} , a horizontal arc of $X_\mathcal{I}^\eta$, or a horizontal boundary arc of \mathcal{R} .) Specifically, let $c_\gamma \subset C_1 \cap \Delta_\varepsilon^\eta$ be a type-1 **sa** with h the associated horizontal arc of statement d. of Lemma 10. Moreover, let $v^\partial \subset H_{\theta\partial}$ be the ∂ -vertical arc that c_γ is adjacent to along with the disc fiber in \mathbf{H} . Now we consider the disc components of $(H_{\theta\partial} \cap \mathcal{T}_\infty) \setminus \Delta_\varepsilon^\eta$. Suppose that Z is a disc-component for which $h \cap H_{\theta\partial} \subset \partial \overline{Z}$. (Notice that there is only one such Z .) If $\partial \overline{Z}$ also contains $x \cap H_{\theta\partial}$ then

x can see the half disc associated with h .

◇

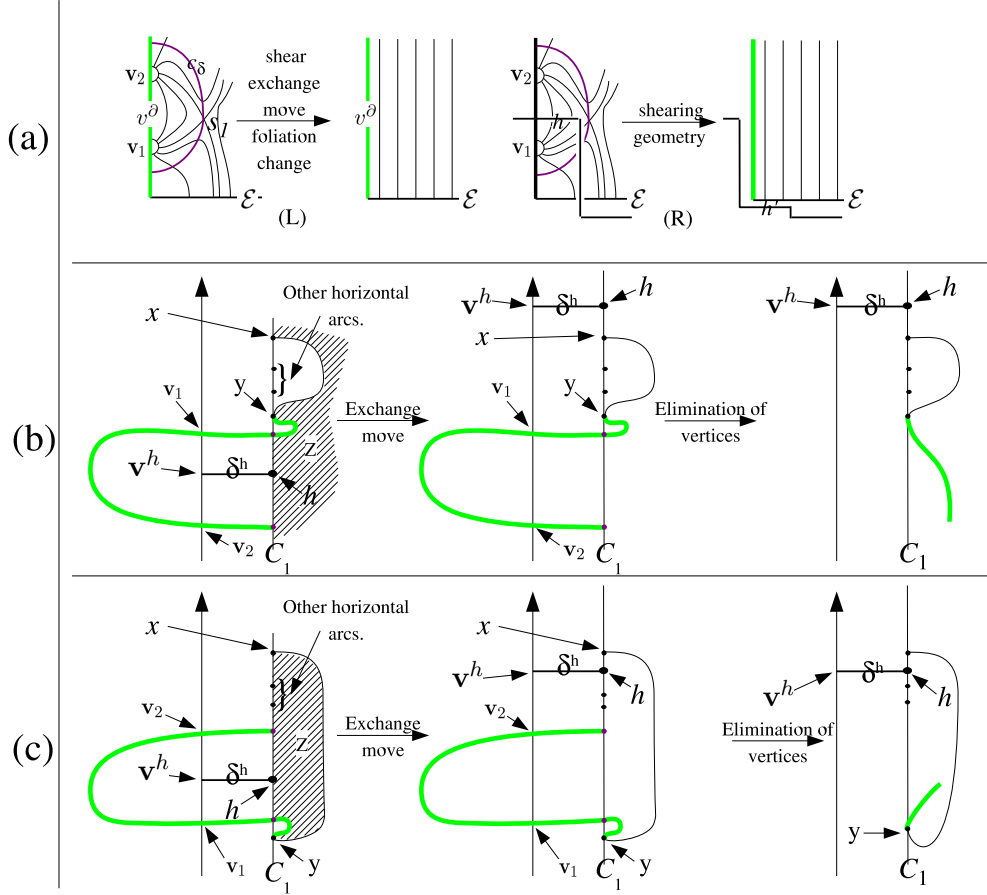


Figure 18: The illustrations should allow for the following interruption. For operation $\mathcal{O}_{3.1}$ we let $x = c_{max}$. For $\mathcal{O}_{3.2}$ we let x be an edgepath in $X_{\mathcal{I}}^\eta$ or a horizontal arc in $\partial\mathcal{R}$. For $\mathcal{O}_{3.3}$ we let x be an edgepath (similar to those in $\mathcal{O}_{2.3}$) containing portions of ∂ -vertical arcs.

$\mathcal{O}_{3.1}$ —Eliminating half discs near c_{max} which c_{max} can see. Let $h \subset X_{\mathcal{I}}^\eta$ be a portion of a horizontal arc associated with a half disc as described in statement c. of Lemma 10. Assume δ^h has an intersection arc $\gamma \subset \delta^h \cap \Delta_\varepsilon^\eta$ which is *innermost* on the component of $\Delta_\varepsilon^\eta \setminus \mathcal{R}$ containing D_{+1} . That is, for every leaf a_θ in the radial foliation of Δ_ε^η intersecting γ we have that when traversing a_θ with \mathbf{v}_{max} as the starting point, the first time a_θ intersects any coning disc is at the point $a_\theta \cap \gamma$. For such a choice of coning disc we apply the transition $(X_{\mathcal{I}}^\eta, \Delta_\varepsilon^\eta) \xrightarrow{T} (X_{\mathcal{I}}^\eta, \Delta_\varepsilon^T)$ to produce our tiled notch disc. This transition necessarily involves positioning the ∂ -vertical arc v^∂ that is adjacent to c_γ so that it intersects \mathbf{A} at (at least) two points.

Referring to Figure 18, this transition yields a tiled foliated subdisc of Δ_ε^η containing \mathbf{v}_{max} with two additional vertices, \mathbf{v}_1 and \mathbf{v}_2 , and a singularity \mathbf{s}_0 . (In Figure 18(a) \mathbf{v}_{max} is at the “top of the page” along with c_{max} and not shown. However, in the Figure 18(b) & (c) cross-

sections the labeled point $x(\subset C_1)$ is to be interrupted as $c_{max} \cap H_{\theta\partial\cdot}$ \mathbf{v}_{max} and \mathbf{v}_2 are positive vertices, whereas \mathbf{v}_1 has negative parity. The **sa** $c_\gamma \subset T\Delta_\varepsilon^\eta \cap C_1$ passes through this singularity and splits off \mathbf{v}_1 and \mathbf{v}_2 . Between \mathbf{v}_1 and \mathbf{v}_2 on the axis \mathbf{A} there is the vertex, \mathbf{v}^h , of coning disc δ^h . And, the b -arcs adjacent to \mathbf{v}_{max} & \mathbf{v}_2 intersect $T\Delta_\varepsilon^\eta \cap C_1$ only at c_{max} & c_γ .

More importantly, we assume that c_{max} can see h (along with c_γ). We illustrate this assumption in the cross-sections of Figure 18(b) & (c). The disc region $Z \subset [(H_{\theta\partial} \cap \mathcal{T}_\infty) \setminus \Delta_\varepsilon^\eta] \subset H_{\theta\partial}$ (shaded regions in figure) has that both $c_{max} \cap \partial\overline{Z}$ and $h \cap \partial\overline{Z}$ are nonempty. (Our cross-sections are in fact the union of the two disc fibers of \mathbf{H} that contain the $v^\partial \subset \partial T\Delta_\varepsilon^\eta$.) Having c_{max} seeing h allows us to eliminate c_γ by a shear horizontal exchange move of h as described next.

We can perform a shear exchange move which will (similar to $\mathcal{O}_{2.3}$) “shear” our coning disc δ^h into two coning disc $\delta^{h'}$ and $\delta^{h''}$. Let $\delta^{h'}$ be the coning disc associated with our **sa** c_γ . Now isotop $\delta^{h'}$ so that the resulting $\mathbf{v}^{h'}$ and \mathbf{v}_{max} are at consecutive horizontal positions. After the isotopy γ and any arcs of $\delta^{h'} \cap C_1$ between $\mathbf{v}^{h'}$ and γ on $\delta^{h'}$ will be eliminated. Thus, the intersection curve c_γ can be eliminated to reduce $[(\Delta_\varepsilon^\eta \setminus \mathcal{R}) \cap [C_1 \setminus (C_1 \cap \mathcal{I})]]$, and we replace $T\Delta_\varepsilon^\eta$ with a radially foliated notch disc Δ_ε^η . (There may be additional **sa**’s of $T\Delta_\varepsilon^\eta \cap C_1$ that are eliminated by this shear exchange move since γ was innermost on Δ_ε^η with respect to \mathbf{v}_{max} but may not be innermost on δ^h with respect to \mathbf{v}^h .)

It is our “seeing” assumption that allows that us to perform our shear horizontal exchange move on h' enabling the elimination of c_γ as illustrated in cross-section sequences. More precisely, throughout all disc fibers in the angular support of c_γ we have that c_{max} can see h and c_{max} . Thus, we have the 3-ball that is the shaded-region- Z cross the angular-support-of- c_γ in which the shear horizontal exchange move on h' can be performed.

Similar to the argument for $\mathcal{O}_{2.1}$ there are two cases depending on the initial ordering of vertices on the axis: $\mathbf{v}_2 < \mathbf{v}^h < \mathbf{v}_1 < \mathbf{v}_{max}$ (Figure 18(b)) and $\mathbf{v}_1 < \mathbf{v}^h < \mathbf{v}_2 < \mathbf{v}_{max}$ (Figure 18(c)). For Figure 18(b) a shear exchange move will position h' so that its horizontal position is just above \mathbf{v}_{max} . Whereas, for Figure 18(c) a shear exchange move will position h' so that its horizontal position is just below \mathbf{v}_{max} .

As in $\mathcal{O}_{2.1}$, our difficulty is in seeing that these two cases in fact correspond to a sequence of vertical exchange moves followed by shear horizontal exchange moves as defined in §1.2. In particular, we need to argue that through the uses of vertical exchange moves the nested feature for the shear horizontal exchange move is satisfied. But, the argument for using vertical exchange moves to move vertical arcs past the angular position of \mathbf{s}_0 (so as to ensure nested) is exactly the same as the argument in $\mathcal{O}_{2.1}$, and we will not repeat it here.

Again, we are not quite done since the isotopy in Figure 16(b) leaves the horizontal position of h' above that of \mathbf{v}_{max} . To resolve this remaining issue we note that the horizontal position of

h' is now maximal and we can perform a first flavor horizontal exchange move on h' making its horizontal position minimal. Notice this will not change the value of $||[\Delta_\varepsilon^\eta \setminus \mathcal{R}] \cap [C_1 \setminus (C_1 \cap \mathcal{I})]||$. Again, we do not forget to place all of the previously moved vertical arcs back into their original position.

We conclude that through a sequence of vertical exchange moves and shear horizontal exchange moves we can eliminate half discs that are associated with innermost intersections of cone discs on the component of $\Delta_\varepsilon^\eta \setminus \mathcal{R}$ containing D_{+1} when c_{max} can see h . The resulting notch disc will have fewer intersection components with $C_1 \setminus \mathcal{I}$. \diamond

$\mathcal{O}_{3.2}$ —Elimination of half disc near edgepath \mathcal{E} that \mathcal{E} can see, away from ∂ -vertical arcs. Again, let $h \subset X_{\mathcal{I}}^\eta$ be a portion of a horizontal arc associated with a half disc, c_γ , as described in statement c. of Lemma 10. Assume the coning disc δ^h has an intersection arc $\gamma \subset \delta^h \cap \Delta_\varepsilon^\eta$ which is *nearest* on a D_0 or the D_{+1} component, $D^* \subset \Delta_\varepsilon^\eta \setminus \text{int}(\mathcal{R})$, to an edge-path $\mathcal{E} \subset \partial D^*$. That is, for every leaf a_θ in the radial foliation of Δ_ε^η intersecting γ we have that it also intersects \mathcal{E} and, when traversing a_θ from $a_\theta \cap \mathcal{E}$ to $a_\theta \cap \gamma$, the first time a_θ intersects any coning disc is at $a_\theta \cap \gamma$. (Again, picture \mathcal{E} having slightly larger angular support than that of c_γ .) Additionally, assume that the only ∂ -vertical arc \mathcal{E} is adjacent to is v^∂ , the ∂ -vertical arc that c_γ is adjacent to. (Thus, $\text{int}(\mathcal{E})$ does not intersect any ∂ -vertical arc.) Finally, assume that \mathcal{E} can see h .

As in $\mathcal{O}_{2.2}$, \mathcal{E} is a portion of either $X_{\mathcal{I}}^\eta \cap D^*$ or a horizontal boundary arc of a component of \mathcal{R} . And, the most straight forward situations are when that \mathcal{E} is a portion of either a horizontal arc of $X_{\mathcal{I}}^\eta$ or a horizontal boundary arc of a component of \mathcal{R} .

We apply the transition $(X_{\mathcal{I}}^\eta, \Delta_\varepsilon^\eta) \xrightarrow{T} (X_{\mathcal{I}}^\eta, \overset{T}{\Delta}_\varepsilon^\eta)$ for this choice of coning disc. Referring to Figure 18, this transition yields a tiled foliated subdisc of $\overset{T}{\Delta}_\varepsilon^\eta$ containing vertices, \mathbf{v}_1 and \mathbf{v}_2 , and a singularity \mathbf{s}_0 . (In Figure 18(a) the edgepath \mathcal{E} in our straight-forward-situation is the labeled horizontal arc. In Figure 18(b) & (c) cross-sections the labeled point $x \subset C_1$ is to be interrupted as $\mathcal{E} \cap H_{\theta\partial}$.) \mathbf{v}_2 and \mathbf{v}_1 are of opposite parity. Again, the **sa** $c_\gamma \subset \overset{T}{\Delta}_\varepsilon^\eta \cap C_1$ passes through this singularity and splits off \mathbf{v}_1 and \mathbf{v}_2 . Between \mathbf{v}_1 and \mathbf{v}_2 on the axis \mathbf{A} there is the vertex, \mathbf{v}^h , of coning disc δ^h . And, the a -arcs adjacent to \mathcal{E} & \mathbf{v}_1 intersect $\overset{T}{\Delta}_\varepsilon^\eta \cap C_1$ only at \mathcal{E} & c_γ .

Again, the important assumption is that \mathcal{E} can see h (along with c_γ). We see this in the cross-sections of Figure 18(b) & (c) when interrupting the labeled point x as $(\mathcal{E} \cap C_1) \cap H_{\theta\partial}$. The disc region $Z \subset [(H_{\theta\partial} \cap \mathcal{T}_\infty) \setminus \Delta_\varepsilon^\eta] \subset H_{\theta\partial}$ has that both $\mathcal{E} \cap \partial \overline{Z}$ and $h \cap \partial \overline{Z}$ are nonempty. Inside the 3-ball that is the shaded-region- Z cross the angular-support-of- c_γ we can perform the shear horizontal exchange move on h' .

We perform a shear exchange move by “shearing” δ^h into two coning disc $\delta^{h'}$ and $\delta^{h''}$. $\delta^{h'}$ will be the coning disc associated with our **sa** c_γ . Now isotop $\delta^{h'}$ so that the resulting $\mathbf{v}^{h'}$ and \mathcal{E} are at consecutive horizontal positions. After the isotopy γ and any arcs of $\delta^{h'} \cap C_1$ between

$\mathbf{v}^{h'}$ and γ on $\delta^{h'}$ will be eliminated. Thus, the intersection curve c_γ can be eliminated to reduce $|\Delta_\varepsilon^\eta \setminus \mathcal{R}| \cap [C_1 \setminus (C_1 \cap \mathcal{I})]|$, and we replace Δ_ε^η with a radially foliated notch disc $\Delta_\varepsilon'^\eta$. (Again, there may be additional **sa** 's of $\Delta_\varepsilon^\eta \cap C_1$ that are eliminated by this shear exchange move since γ was innermost on Δ_ε^η with respect to \mathbf{v}_{max} but may not be innermost on δ^h with respect to \mathbf{v}^h .)

As in $\mathcal{O}_{2.2}$ and $\mathcal{O}_{3.1}$, our difficulty is in seeing that this isotopy of h' corresponds to a sequence of vertical exchange moves followed by shear horizontal exchange moves as defined in §1.2. The argument for using vertical exchange moves to move vertical arcs past the angular position of \mathbf{s}_0 (so as to ensure nested) is exactly the same as the argument in $\mathcal{O}_{2.2}$. The value of $|\Delta_\varepsilon^\eta \setminus \mathcal{R}| \cap [C_1 \setminus (C_1 \cap \mathcal{I})]|$ is reduced with the elimination of c_γ . Again, we do not forget to place all of the previously moved vertical arcs back into their original position.

The remaining case is when \mathcal{E} is in $X_\mathcal{I}^\eta$ but is not a portion of a single horizontal arc. As in $\mathcal{O}_{2.2}$, we can transform \mathcal{E} into a portion of such a horizontal arc by utilizing $\mathcal{O}_{1.1}$ and imitating the argument in Step-1 & Step-2 of $\mathcal{O}_{1.2}$. Such a transformation of \mathcal{E} will necessarily employ the use of horizontal and vertical exchange moves, and horizontal simplifications.

We conclude that through a sequence of vertical exchange moves and shear horizontal exchange moves we can eliminate half discs in this situation. \diamond

$\mathcal{O}_{3.3}$ –Elimination of half disc near edgepath \mathcal{E} that \mathcal{E} can see, adjacent to ∂ -vertical arcs. Our assumptions are exactly as those in $\mathcal{O}_{3.2}$ except we allow for \mathcal{E} to be adjacent to ∂ -vertical arcs that c_γ is not adjacent to. We list the initial possibilities: (i) Δ_e^η and $\text{int}(\mathcal{E})$ intersects one containing ∂ -vertical arcs; and, (ii) Δ_f^η and $\text{int}(\mathcal{E})$ intersects with one or two ∂ -vertical arcs.

The argument is a combining of the arguments of $\mathcal{O}_{2.3}$ and $\mathcal{O}_{3.2}$. The details at this points should be readily accessible to the reader. We concluded that through a sequence of elementary moves half discs in this situation can be eliminated. Moreover, the count $|\Delta_\varepsilon^\eta \setminus \mathcal{R}| \cap [C_1 \setminus (C_1 \cap \mathcal{I})]|$ is reduced. \diamond

\mathcal{O}_4 –Using shear vertical simplification on product of subdisc fibers. We generalize the idea of using virtual rectangular discs to perform shear vertical exchange moves. We assume that the foliation of $\Delta_\varepsilon^\eta \setminus (\Delta_\varepsilon^\eta \cap C_1)$ contains no whole or half discs. Let $[\theta_1, \theta_2] \subset S^1$ be an angular interval such that H_{θ_1} and H_{θ_2} are disc fibers of \mathbf{H} containing ∂ -vertical arcs of $\partial\Delta_\varepsilon^\eta$ having differing edge assignments. To list the obvious cases: (A) there is a back edge, v_1^∂ , having position θ_1 and a front edge, v_2^∂ having position θ_2 ; or, (B) there is a front edge, v_1^∂ , having position θ_1 and a back edge, v_2^∂ , having position θ_2 .

We assume that over the angular interval $[\theta_1, \theta_2]$ our $\partial\Delta_\varepsilon^\eta$ contains no vertical arcs. If we have case (A) (respectively, case (B)) then any **sa** having endpoints on v_1^∂ and v_2^∂ with angular support $[\theta_1, \theta_2]$ will be a type-2 (respectively, type-4).

We consider the components of $[\cup_{\theta \in (\theta_1, \theta_2)} H_\theta \cap \mathcal{T}_\infty] \setminus \Delta_\varepsilon^\eta$. The reader should realize that these components are topological open 3-ball. We wish to distinguish one of these 3-ball components. In particular, \hat{B} will be the 3-ball component whose closure contains the horizontal subarcs of $X_T^\eta \setminus \partial \Delta_\varepsilon^\eta$ whose endpoints correspond to the endpoints of v_1^∂ and v_2^∂ .

To expand on this terse description, we have case (B) (respectively, case (A)) as follows. Referring back to Figure 8 we observe that the left ∂ -vertical arc is a front edge with its “bottom of the page” endpoint (respectively, “top of the page” endpoint) having the braid ‘entering’ (respectively, ‘exiting’) $\partial \Delta_\varepsilon^\eta$. The right ∂ -vertical arc is a back edge with the braid ‘exiting’ (respectively, ‘entering’) $\partial \Delta_\varepsilon^\eta$ at the “top of the page” (respectively, “bottom of the page”) endpoint. \hat{B} is the component whose closure contains the portion of the braid entering and exiting (respectively, exiting and entering) $\partial \Delta_\varepsilon^\eta$.

For both case (A) and (B) it should be clear that each 3-ball component has a natural product structure, $D \times (\theta_1, \theta_2)$ where D is an open disc component of $(H_\theta \cap \mathcal{T}_\infty) \setminus \Delta_\varepsilon^\eta \subset H_\theta$ for $\theta \in (\theta_1, \theta_2)$. In particular, for any 3-ball component $D \times (\theta_1, \theta_2)$ which is not \hat{B} , if $v \subset \overline{D \times (\theta_1, \theta_2)}$ is a vertical arc of X_T^η having angular position nearest to, say, θ_1 , then we have a virtual rectangular disc $R = v \times [\theta_1, \theta_2] \subset \overline{D \times (\theta_1, \theta_2)}$. Moreover, $R \cap \Delta_\varepsilon^\eta = \emptyset$ and we can use R to push v out of $\overline{D \times (\theta_1, \theta_2)}$. As with our previous use of virtual rectangular discs, this push will correspond to a sequence of vertical exchange moves followed by a single shear vertical simplification. \diamond

3 Proof of Theorem 4.

With the operations of §2.4 in place we are now in a position to prove Proposition 7 and, thus, Theorem 4. We start by enhancing our complexity measure on X_T^η so as to incorporate information from our notch disc. Specifically, we define the lexicographically ordered complexity $\mathcal{C}(X_T^\eta, \Delta_\varepsilon^\eta) = (n_1, n_2, n_3)$ where: $n_1 = \mathcal{C}(X_T^\eta)$; n_2 is the number of components of $[\Delta_\varepsilon^\eta \setminus \mathcal{R}] \cap [C_1 \setminus (C_1 \cap \mathcal{I})]$; and, n_3 is the number of vertical arcs in $\partial \Delta_\varepsilon^\eta$. Then we will have an obvious destabilizing disc when $\mathcal{C}(X_T^\eta, \Delta_d^\eta) = (-, 0, 1)$; an obvious exchange disc when $\mathcal{C}(X_T^\eta, \Delta_e^\eta) = (-, 0, 0)$; and, an obvious flying disc when $\mathcal{C}(X_T^\eta, \Delta_f^\eta) = (-, 0, 1)$. Ideally, for an obvious disc all of the vertically arcs not in $\partial \Delta_\varepsilon^\eta$ have been pushed into \mathcal{I} by shear-vertical simplification. Thus, although not necessary for recognizing an obvious disc, the absolute minimal complexity has: $\mathcal{C}(X_T^\eta, \Delta_d^\eta) = (2, 0, 1)$; $\mathcal{C}(X_T^\eta, \Delta_e^\eta) = (0, 0, 0)$; and, $\mathcal{C}(X_T^\eta, \Delta_f^\eta) = (1, 0, 1)$.

Now, let X be a closed braid which admits one of our three braid isotopies and let $X \xrightarrow{N} X^\eta$ be a transition to an arc presentation. Since associated to X there was a disc Δ_ε depicting a braid isotopy there exists a Δ_ε^η associated to X^η also depicting the same braid isotopy. We can apply Lemma 8 without changing $\mathcal{C}(X^\eta)$. The position of the ∂ -vertical arcs of Δ_ε^η gives us the

existence and position of our shearing intervals \mathcal{I} . We can then apply Lemmas 9, 10 and 11 so as to place Δ_ε^η into a position for calculating the values of our 3-tuple complexity measure $\mathcal{C}(X_\mathcal{I}^\eta, \Delta_\varepsilon^\eta)$ pair.

The argument for achieving minimal complexity should be clear. Applying a sequence of \mathcal{O}_2 and \mathcal{O}_3 operations we can eliminate all whole and half discs while reducing n_2 and, thus, $\mathcal{C}(X_\mathcal{I}^\eta, \Delta_\varepsilon^\eta)$. Specifically, if a half disc is near an edgepath that is either a portion of $X_\mathcal{I}^\eta$ or a horizontal boundary arc of a component of \mathcal{R} then the edgepath can see the horizontal arc associated with the associated **sa**. So starting with **scc** that are close to c_{max} or near an edgepath; or **sa** that in near and can be seen by an edgepath or c_{max} , we can iterate these two operations until every **scc** and **sa** is eliminated.

Next applying the appropriate \mathcal{O}_1 sub-operation we can simplify the middle boundary of Δ_ε^η until n_3 is either 0 (for Δ_e^η) or 1 (for Δ_d^η or Δ_f^η). Along the way we may have applied some number of horizontal, vertical and shear vertical simplifications to reduce $n_1 = \mathcal{C}(X_\mathcal{I}^\eta)$. Since all of our \mathcal{O} -operations involve the use of elementary moves the four statements of Proposition 7 are satisfied.

We pause briefly to discuss the nature of the finiteness of our complexity measure. In view of Remarks 1, 2 and 3 plus the “connect the dots” discussion just prior to the statement of Theorem 4, for a specified \mathcal{I} and middle boundary edge path there may be infinitely many possible ∂ -vertical arcs yielding infinitely many possible Δ_ε^η . Thus, an initial value of n_2 can be arbitrarily large. However, initial values of n_1 and n_3 are bounded. Moreover, for any values of n_1 and n_3 there are only finitely many possible cyclic orderings of horizontal and vertical arcs. Thus, within this finite collection of orderings there will exists some Δ_ε^η having its associated n_2 being zero. With an $n_2 = 0$ we can reduce n_3 to its associated minimal value using \mathcal{O}_1 operations.

Having established Proposition 7 we have also established statements 1 and 2 of Theorem 4. In order to establish statement 3 of Theorem 4 we need to think about how the rectangles of \mathcal{R} alter the $X_\mathcal{I}'^\eta$ of statement 3 and, thus, the standard projection of a X' coming from $X_\mathcal{I}'^\eta \xrightarrow{B} X'$.

It is simplest to deal with each braid isotopy individually in turn.

A destabilizing disc—We briefly summarize the salient features of the pair $(X_\mathcal{I}'^\eta, \Delta_d'^\eta)$ (with $\mathcal{C}(X_\mathcal{I}'^\eta, \Delta_d'^\eta) = (-, 0, 1)$) that established the validity of Proposition 7 for a destabilization. Since $\partial\Delta_d'^\eta$ has a single ∂ -vertical arc we know that there are no type-2, -3, or -4 **sa** in $\Delta_d'^\eta \cap C_1$. In fact, as a result of the simplification in $\mathcal{O}_{1,2}$ we know that:

1. $\text{int}(\Delta_d'^\eta) \cap C_1 = c_{max}$.
2. $\partial\Delta_d'^\eta = h_1 \dot{\cup} v^\partial \dot{\cup} h_2 \dot{\cup} v_2 \dot{\cup}$.

3. If Δ_d^η corresponds to a positive (respectively, negative) destabilization then v^∂ is a front (respectively, back) edge, and h_2 (respectively, h_1) has maximal horizontal position over all horizontal arcs of $X_\mathcal{T}^\eta$.
4. Recalling the decomposition $\Delta_d^\eta = D_{+1} \cup_{c_{max}} N$ where $\partial N = c_{max} \cup \partial \Delta_d^\eta$ we have that $N = R_1 \cup R_2$ where:
 - a. R_1 has the same angular support as h_1 ($[\theta_2, \theta^\partial]$) and vertical support $[z^{h_1}, z_{max}]$.
 - b. R_2 has the same angular support as h_2 ($[\theta^\partial, \theta_2]$) and the vertical support $[z^{h_2}, z_{max}]$.

There are the positive and negative destabilizing cases to argue. The arguments are similar, so we will present the positive destabilization case and leave the details of the negative case to the reader.

The existence of N divides \mathcal{T}_∞ up into three potential regions where braiding of $X_\mathcal{T}^\eta \setminus \partial \Delta_d^\eta$ can occur: \mathcal{H}_1 is over the interval $[\theta^\partial, \theta_2]$ and below z^{h_2} ; \mathcal{H}_2 is over the interval $[\theta_2, \theta^\partial]$ and above z^{h_1} ; and, \mathcal{H}_3 is over the interval $[\theta_2, \theta^\partial]$ and below z^{h_1} . We will achieve a standard projection easily seen as equivalent (through type-III Reidemeister moves) to $W\sigma_{n-1}$ when: \mathcal{H}_3 contains no vertical arcs, thus no braiding; and \mathcal{H}_2 contains a single vertical arc which will be adjacent to h_2 .

Having no vertical arcs in \mathcal{H}_3 is easily achieved by vertical exchange moves. Specifically, every vertical arc in \mathcal{H}_3 has vertical support either below or above z^{h_1} . Thus, using the idea in Step-3 of $\mathcal{O}_{1.3}$ for constructing virtual rectangular discs, through a sequence of vertical exchange moves we can push all vertical arcs out of \mathcal{H}_3 and into \mathcal{H}_1 .

Finally, we consider the vertical arcs in \mathcal{H}_2 . Since z^{h_2} is maximal, there must be a vertical arc $v' \subset X_\mathcal{T}^\eta$ having angular position in $(\theta_2, \theta^\partial)$ and adjacent to h_2 . Let θ' be it angular position. Then repeating the vertical exchange moves arguments of Step-3 of $\mathcal{O}_{1.2}$ we can construct virtual rectangular discs that are in the complement of Δ_d^η for pushing all the vertical arcs in \mathcal{H}_2 having angular position in $(\theta', \theta^\partial)$ forward past v^∂ , and in \mathcal{H}_2 having angular position in (θ_2, θ') backward past v_2 . All such pushes move vertical arcs into \mathcal{H}_1 .

All of these vertical exchange moves do no change the complexity measure of our pair $(X_\mathcal{T}^\eta, \Delta_d^\eta)$. Thus, we have established statement 3 of Theorem 4 for a destabilization.

An exchange disc—We consider a pair $(X_\mathcal{T}^\eta, \Delta_e^\eta)$ with $\mathcal{C}(X_\mathcal{T}^\eta, \Delta_e^\eta) = (-, 0, 0)$ which establishes the validity of Proposition 7 for an exchange move. Since $\partial \Delta_e^\eta$ has a two ∂ -vertical arcs of differing edge assignment we know that there are only type-2 and -4 **sa** in $\Delta_e^\eta \cap C_1$. From our $\mathcal{O}_{1.3}$ and $\mathcal{O}_{1.4}$ operations we know that it is useful to breakup our discussion into two cases: $\mathcal{R} = \emptyset$ and $\mathcal{R} \neq \emptyset$. When $\mathcal{R} = \emptyset$ we know from $\mathcal{O}_{1.3}$ that:

1. $\Delta_e^\eta \cap C_1 = c_{max} \cup \partial \Delta_e^\eta$.

2. $\partial\Delta_e^\eta = h_1 \dot{\cup} v_1^\partial \dot{\cup} h_2 \dot{\cup} v_2^\partial \dot{\cup}$. For convenience, we say v_1^∂ is a back edge and v_2^∂ is a front edge with $z^{h_2} < z^{h_1}$.
3. Recalling the decomposition $\Delta_e^\eta = D_{+1} \cup_{c_{max}} N$ where $\partial N = c_{max} \cup \partial\Delta_e^\eta$ we have that $N = R_1 \cup R_2$ where:
 - a. R_1 has the same angular support as h_1 ($[\theta_2^\partial, \theta_1^\partial]$) and vertical support $[z^{h_1}, z_{max}]$.
 - b. R_2 has the same angular support as h_2 ($[\theta_1^\partial, \theta_2^\partial]$) and the vertical support $[z^{h_2}, z_{max}]$.

The decomposing disc $R_1, R_2 \subset N$ divides \mathcal{T}_∞ up into four potential regions where braiding of $X_\mathcal{T}^\eta \setminus \partial\Delta_d^\eta$ can occur: \mathcal{H}_1^+ is over the interval $[\theta_2^\partial, \theta_1^\partial]$ and above z^{h_1} (and under R_1 in \mathcal{T}_∞); \mathcal{H}_1^- is over the interval $[\theta_2^\partial, \theta_1^\partial]$ and below z^{h_1} (and below R_1); \mathcal{H}_2^+ is over the interval $[\theta_1^\partial, \theta_2^\partial]$ and above z^{h_2} (and under R_2 in \mathcal{T}_∞); and, \mathcal{H}_2^- is over the interval $[\theta_1^\partial, \theta_2^\partial]$ and below z^{h_2} (and below R_2). We will achieve a standard projection for X' easily seen as having the needed braid word structure WU (W a word in \mathcal{W}^t and U a word in \mathcal{U}^s with $s \leq t$) when \mathcal{H}_1^+ and \mathcal{H}_2^- contains no vertical arcs, thus no braiding. The number of horizontal arcs of $X_\mathcal{T}^\eta$ at the angle, say θ_1^∂ having horizontal position in the interval (z^{h_2}, z^{h_1}) corresponds to $t - s + 2$.

For two angular intervals, $[\theta_2^\partial, \theta_1^\partial]$ and $[\theta_1^\partial, \theta_2^\partial]$, we notice that we are in the situation described in \mathcal{O}_4 . Moreover, for $[\theta_2^\partial, \theta_1^\partial]$ (respectively, $[\theta_1^\partial, \theta_2^\partial]$) we have that \mathcal{H}_1^- (respectively, \mathcal{H}_2^+) corresponds to the distinguish \hat{B} component. Thus, by \mathcal{O}_4 we can push all vertical arcs in \mathcal{H}_1^+ (respectively, \mathcal{H}_2^-) into \mathcal{H}_2^+ (respectively, \mathcal{H}_1^-) using a sequence of vertical exchange moves.

Proceeding to the case where $\mathcal{R} \neq \emptyset$ we acknowledge that using only the collection of elementary moves on arc presentations with shearing intervals, it may not be possible to place $X_\mathcal{T}^\eta$ in a position such that for $X_\mathcal{T}^\eta \xrightarrow{B} X'$ we have $\beta(X') = WU$ (where $W \in \mathcal{W}^t$ and $U \in \mathcal{U}^s$, $s \leq t$). To be clear, it is readily seen that by allowing additional moves on the our shear arc presentations that correspond to the inverse of horizontal simplification we can produce presentations $X_\mathcal{T}^{\eta\eta}$ such that $\mathcal{R} = \emptyset$ and, by the above argument, for $X_\mathcal{T}^{\eta\eta} \xrightarrow{B} X''$ we have the desired braid word form $\beta(X'') = WU$. Unfortunately, we lose the feature that our complexity measure is non-increasing.

However, referring back to Remarks 1 and 2 we note: first, since $X_\mathcal{T}^{\eta\eta}$ admits an exchange move, it admits a thin exchange move; and, second, there are infinitely many exchange move discs associated with a thin exchange move of $X_\mathcal{T}^{\eta\eta}$. In particular, from Remark 2 we have the following.

Let X'' be an n -braid such that $\beta(X'') = WU$, where $W \in \mathcal{W}^t$ and $U \in \mathcal{U}^s$ with $t - s$ minimal, i.e. a thin exchange move. Then for every integer pair (p, q) the n -braid associated with the word

$$W'U' = [\tau_{[1, s-1]}^p W \tau_{[1, s-1]}^{-p}] [\tau_{[t+2, n]}^q U \tau_{[t+2, n]}^{-q}]$$

admits the same thin exchange move. Referring the reader back to §2.1 and our description of exchange-move-disc/braid-presentation, we realize that for the closed n -braid X there are infinitely many exchange discs $\Delta_e(p, q)$ satisfying conditions E-a through E-e with $\partial\Delta_e(p, q) = \alpha_{h_1} \dot{\cup} [\alpha_{v_1}^\partial(p, q)] \dot{\cup} \alpha_{h_2} \dot{\cup} [\alpha_{v_2}^\partial(p, q)] \dot{\cup}$, i.e. all such discs have the same two horizontal arcs, but differing ∂ -vertical arcs indexed by (p, q) . Shifting this picture over to exchange-move-disc/arc-presentation, we realize that there are infinitely many notch exchange discs, $\Delta_e^\eta(p, q)$, satisfying conditions EA-a through EA-e. Moreover, $\partial\Delta_e^\eta(p, q) = h_1 \dot{\cup} [v_1^\partial(p, q)] \dot{\cup} h_2 \dot{\cup} [v_2^\partial(p, q)] \dot{\cup}$. That is, the boundary all such discs have the same two horizontal arcs, but differing ∂ -vertical arcs indexed by (p, q) .

Using the radial foliations of $\Delta_e(p, q)$ we take note of the following *winding behavior*. As in condition E-a we have $\alpha_{v_1}^\partial(p, q) \subset H_{\theta_1^\partial}$ and $\alpha_{v_2}^\partial(p, q) \subset H_{\theta_2^\partial}$. So in the interval $(\theta_1^\partial, \theta_2^\partial)$ there are, say, $s - 1$ trivial strands of X'' ; and, in $(\theta_2^\partial, \theta_1^\partial)$ there are $n - (t + 1)$ trivial strands of X'' . Moreover, each leaf $a = \Delta_e(p, q) \cap H_\theta$, $\theta \in (\theta_1^\partial, \theta_2^\partial)$ winds p times around these $s - 2$ trivial strands as seen in H_θ —one strand corresponds to the boundary of $\Delta_e(p, q)$. (This is winding in the sense that any ray in H_θ starting at the point corresponding to where the strand intersects H_θ must algebraically intersect the leaf a a minimal of p times.) Similarly, each leaf $a = \Delta_e(p, q) \cap H_\theta$, $\theta \in (\theta_2^\partial, \theta_1^\partial)$ winds q times around these $n - (t + 2)$ trivial strands as seen in H_θ .

This winding behavior extends to all braid representatives in $\mathcal{B}_n(X'')$. Specifically, for any fixed representative $X \in B_n(X'')$, since X is braid isotopic to X'' there is an ambient isotopy of $S^3 \setminus \mathbf{A}$ taking the family $\Delta_e(p, q)$ to exchange discs for X . Moreover, for X there must exist a pair (P_X, Q_X) such that for $P_X < |p|$ and $Q_X < |q|$ we have two similar statements of *winding behavior*:

- $(\theta_1^\partial, \theta_2^\partial)$ As $|p|$ increases, each leaf $a = \Delta_e(p, q) \cap H_\theta$, $\theta \in (\theta_1^\partial, \theta_2^\partial)$ increasingly winds (geometrically) around these $s - 1$ strands as seen in H_θ .
- $(\theta_2^\partial, \theta_1^\partial)$ As $|q|$ increases, each leaf $a = \Delta_e(p, q) \cap H_\theta$, $\theta \in (\theta_2^\partial, \theta_1^\partial)$ increasingly winds (geometrically) around these $n - (t + 1)$ strands as seen in H_θ .

Now taking a transition $X \xrightarrow{N} X^\eta$ we see that we have these two winding statements also for $\Delta_e^\eta(p, q)$ with $P_X < |p|$ and $Q_X < |q|$.

Next, consider an infinite collection of pairs $(X_{\mathcal{I}}^\eta, \Delta_e^\eta(p, q))$ which are associate as above to a thin exchange move. In particular, for any two 2-tuples, (p, q) and (p', q') , we have $\partial\Delta_e^\eta(p, q) \cap X_{\mathcal{I}}^\eta = \partial\Delta_e^\eta(p', q') \cap X_{\mathcal{I}}^\eta$. For any fixed (p, q) we can apply our \mathcal{O} -operations to reduce the complexity measure for the shear-arc-presentation/notch-disc pair to $(-, 0, 0)$. Now recall the observation just prior to Theorem 4 that there are only finitely many shear arc presentations resulting from applying a sequence of elementary moves to $X_{\mathcal{I}}^\eta$. This observation implies that

there exists a resulting $X_{\mathcal{T}}'^{\eta}$ and an infinite set of 2-tuples, $\mathcal{S} \subset \mathbb{Z} \times \mathbb{Z}$ such that:

1. the values $|p|$ & $|q|$ are unbounded over all $(p, q) \in \mathcal{S}$;
2. $h_1 \cup h_2 = X_{\mathcal{T}}'^{\eta} \cup \partial \Delta_e'^{\eta}(p, q)$ for all $(p, q) \in \mathcal{S}$;
3. $\mathcal{C}(X_{\mathcal{T}}'^{\eta}, \Delta_e'^{\eta}(p, q)) = (-, 0, 0)$ for all $(p, q) \in \mathcal{S}$.

The reader should realize that increasing the values of $|p|$ & $|q|$ causes an increase in the number of type -2 & -4 **sa** 's in $\Delta_e'^{\eta}(p, q) \cap C_1$ and, thus, the value of $|\mathcal{R}|$. Our immediate goal is to show that $X_{\mathcal{T}}'^{\eta}$ is “close” to being a shear arc presentation that transitions to a closed braid which illustrates an obvious exchange move. By “close” we will mean that we may have to apply some number of horizontal/shear horizontal exchange moves.

To achieve this stated goal we proceed by understanding the implications of the winding behavior in the radial foliation of our exchange discs. We know that for large enough values $|p_0|$ and $|q_0|$, $(p_0, q_0) \in \mathcal{S}$, the radial leaves of $\Delta_e'^{\eta}(p_0, q_0)$ will wind around a maximal number (due to having a thin exchange move) of strands of $X_{\mathcal{T}}'^{\eta}$ within the specified angular intervals. This winding implies the existence of a subdisc $D_{\theta}^w \subset H_{\theta}$ for $\theta \in (\theta_1^{\partial}, \theta_2^{\partial})$ (respectively, $\theta \in (\theta_2^{\partial}, \theta_1^{\partial})$) which satisfies the following.

- a. $\partial D_{\theta}^w = e_{\theta}^1 \dot{\cup} e_{\theta}^2 \dot{\cup}$ where $e_{\theta}^1 \subset \Delta_e'^{\eta}(p_0, q_0) \cap H_{\theta}$ and $e_{\theta}^2 \subset C_1 \cap H_{\theta}$ with $\theta \in (\theta_1^{\partial}, \theta_2^{\partial})$ (respectively, $\theta \in (\theta_2^{\partial}, \theta_1^{\partial})$);
- b. $e_{\theta}^1 \cap e_{\theta}^2 = \partial e_{\theta}^1 = \partial e_{\theta}^2$;
- c. $D_{\theta}^w \cap X_{\mathcal{T}}'^{\eta} = \text{int}(D_{\theta}^w) \cap X_{\mathcal{T}}'^{\eta}$;
- d. $|D_{\theta}^w \cap X_{\mathcal{T}}'^{\eta}| = s - 1$ (respectively, $|D_{\theta}^w \cap X_{\mathcal{T}}'^{\eta}| = n - (t + 1)$). That is, D_{θ}^w intersects a maximal number of strands of $X_{\mathcal{T}}'^{\eta}$ since $\Delta_e'^{\eta}(p_0, q_0)$ is associated with a thin exchange move.

Since $\mathcal{C}(X_{\mathcal{T}}'^{\eta}, \Delta_e'^{\eta}(p_0, q_0)) = (-, 0, 0)$, we have the product structure discussed in \mathcal{O}_4 inside the 3-ball $B^{1,2} = \cup_{\theta \in (\theta_1^{\partial}, \theta_2^{\partial})} D_{\theta}^w$ (or, $B^{2,1} = \cup_{\theta \in (\theta_2^{\partial}, \theta_1^{\partial})} D_{\theta}^w$) that allow us to assume after performing some number of shear vertical simplifications that this 3-ball contains no vertical arcs of $X_{\mathcal{T}}'^{\eta}$. Similarly, the bounded 3-ball components of $\cup_{\theta \in (\theta_1^{\partial}, \theta_2^{\partial})} (H_{\theta} \cap \mathcal{T}_{\infty}) \setminus B^{1,2}$ have product structures that allow us to apply \mathcal{O}_4 removing vertical arcs. There is still the possibility that there are horizontal arcs intersecting the boundary of the closure of these 3-balls components. We will shortly deal with this possibility. (We have a similar statement for $B^{2,1}$ and interval $(\theta_2^{\partial}, \theta_1^{\partial})$. However, if $B^{1,2}/B^{2,1}$ are simple enough there may be no such unbounded complementary 3-balls components.)

Now consider the rectangular shaped disc components of

$$[\cup_{\theta \in (\theta_1^\partial, \theta_2^\partial)} (C_1 \cap H_\theta)] \setminus B^{1,2}.$$

We observe that no component contains horizontal subarc of $X_{\mathcal{T}}^\eta$ having angular support containing $(\theta_1^\partial, \theta_2^\partial)$. Since, if such a horizontal subarc occurred it would contradict our assumption that D_θ^w intersected a maximal number of trivial strands for $\theta \in (\theta_1^\partial, \theta_2^\partial)$, i.e. a contradiction of our thin exchange move assumption. We can make a similar statement for components of

$$[\cup_{\theta \in (\theta_2^\partial, \theta_1^\partial)} (C_1 \cap H_\theta)] \setminus B^{2,1}.$$

We observe that, after performing some number of horizontal exchange moves to ensure that the horizontal position of all horizontal arcs intersecting (say) $B^{2,1}$ are above the horizontal position of all the horizontal arcs intersecting $B^{1,2}$ we can assume that the vertical support of $B^{1,2}$ and $B^{2,1}$ are disjoint. (Since we now have $B^{1,2}$ and $B^{2,1}$ containing no vertical arcs our necessarily nested condition is easily satisfied for horizontal exchange moves.) Moreover, over the vertical support of $B^{1,2}$ (respectively, $B^{2,1}$) within the angular interval $(\theta_1^\partial, \theta_2^\partial)$ (respectively, $(\theta_2^\partial, \theta_1^\partial)$) we have that the only horizontal arcs are those contained in $B^{1,2}$ (respectively, $B^{2,1}$).

Finally, for the angular interval $(\theta_1^\partial, \theta_2^\partial)$ (respectively, $(\theta_2^\partial, \theta_1^\partial)$) if there are any horizontal arcs below (respectively, above) the vertical support of $B^{1,2}$ (respectively, $B^{2,1}$) then, through a sequence of horizontal exchange moves and shear horizontal exchange moves, we can move them above (respectively, below) $B^{1,2}$ (respectively, $B^{2,1}$). (Again, the nested condition necessary for the performance of these exchange move is readily seen to be satisfied.) For the resulting $X_{\mathcal{T}}^\eta$ it should be apparent that for $X_{\mathcal{T}}^\eta \xrightarrow{B} X'$ we have $\beta(X')$ admitting an exchange move.

A flying disc—As with our previous two isotopies we consider a pair $(X_{\mathcal{T}}^\eta, \Delta_f^\eta)$ with $\mathcal{C}(X_{\mathcal{T}}^\eta, \Delta_f^\eta) = (-, 0, 1)$ for an elementary flype move. $\partial \Delta_f^\eta$ has a three ∂ -vertical arcs with edge assignments front-back-front (respectively, back-front-back) for a positive (respectively, negative) flype. We will argue the positive flype and leave the details of the negative flype to the reader.

From our $\mathcal{O}_{1.4}$, $\mathcal{O}_{1.5}$ and $\mathcal{O}_{1.6}$ operations we know that it is useful to breakup our discussion into two cases: $\mathcal{R} = \emptyset$ and $\mathcal{R} \neq \emptyset$. When $\mathcal{R} = \emptyset$ we know from $\mathcal{O}_{1.5}$ that:

1. $\Delta_f^\eta \cap C_1 = c_{max} \cup \partial \Delta_f^\eta$.
2. $\partial \Delta_f^\eta = h_1 \dot{\cup} v_1^\partial \dot{\cup} h_2 \dot{\cup} v_2^\partial \dot{\cup} h_3 v_3^\partial \dot{\cup} h_4 \dot{\cup} v_4 \dot{\cup}$. For convenience, we say v_1^∂ & v_3^∂ are front edges and v_2^∂ is a back edge with z^{h_1} minimal and $z^{h_3} < z^{h_2}$.
3. Recalling the decomposition $\Delta_f^\eta = D_{+1} \cup_{c_{max}} N$ where $\partial N = c_{max} \cup \partial \Delta_f^\eta$ we have that $N = R_1 \cup R_2 \cup R_3$ where:
 - a. R_1 has the same angular support as h_1 ($[\theta^{v_4}, \theta_1^\partial]$) and vertical support $[z^{h_1}, z_{max}]$.

- b. R_2 has the same angular support as h_2 ($[\theta_1^\partial, \theta_2^\partial]$) and the vertical support $[z^{h_2}, z_{max}]$.
- c. R_3 has the same angular support as h_3 ($[\theta_2^\partial, \theta_3^\partial]$) and the vertical support $[z^{h_3}, z_{max}]$.
- d. R_4 has the same angular support as h_4 ($[\theta_3^\partial, \theta^{v_4}]$) and the vertical support $[z^{h_4}, z_{max}]$.

We emulate the \mathcal{H}_i^\pm -analysis in the exchange move situation. In particular, we have N dividing \mathcal{T}_∞ up into eight potential regions where braiding of $X_{\mathcal{I}}^\eta \setminus \partial\Delta_d^\eta$ can occur: \mathcal{H}_i^+ is over the interval corresponding to the angular support of h_i and above z^{h_i} ; and, \mathcal{H}_i^- is over the interval corresponding to the angular support of h_i and below z^{h_i} , for $i \in \{1, 2, 3, 4\}$. $\mathcal{H}_1^+, \mathcal{H}_2^+, \mathcal{H}_3^+, \mathcal{H}_4^+$ will be in \mathcal{T}_∞ under the discs R_1, R_2, R_3, R_4 , respectively.

Through the application of vertical exchange moves and shear horizontal exchange moves in \mathcal{H}_1^- we obtain our assumption of statement 2 that z^{h_1} is minimal. (The assumption that $z^{h_3} < z^{h_2}$ will be dealt with shortly.) Similarly, after some number of shear vertical exchange moves to move any vertical arcs in \mathcal{H}_4^- past v_3^∂ we can perform shear horizontal exchange moves so that, in the angular interval $(\theta_3^\partial, \theta^{v_4})$, z^{h_4} is two horizontal positions above z^{h_1} .

Now, to determine whether the resulting $X_{\mathcal{I}}^\eta$ has a transition $X_{\mathcal{I}}^\eta \xrightarrow{B} X'$ with $\beta(X')$ corresponding to a word admitting an elementary flype, we need to inspect the horizontal position of h_2 to see if it is two above minimal over its angular support, and if the horizontal position of h_3 is minimal over its angular support. (We will deal with the unsatisfactory nature of having to inspect momentarily.)

If the horizontal positions of the boundary are correct then we will have braid blocks occurring as follows (see Figure 9 FA^+):

- W_1 – A braid block in \mathcal{H}_1^+ that is under the $R_1(\subset N)$ disc. Since there are no available strands in \mathcal{H}_1^- there is no braiding.
- U – A braid block in \mathcal{H}_2^- . The horizontal position of h_2 makes this a two strand braid block. Using \mathcal{O}_4 we can push all vertical arc in \mathcal{H}_2^+ that are under R_2 past either v_1^∂ or v_2^∂ .
- W_2 – A braid block in \mathcal{H}_3^+ that is under the R_3 disc. Since there are no available strands in \mathcal{H}_3^- there is no braiding.
- \emptyset – There is no braiding in \mathcal{H}_4^+ under R_4 since any vertical arcs can be pushed past v_4 into \mathcal{H}_1^+ . And, there is no braiding in \mathcal{H}_4^- since there is but one strand.

We finally address the issue of having to inspect the horizontal positioning of h_2 and h_3 . As in the exchange move situation, Remark 2 gives us that $X_{\mathcal{I}}^\eta$ has infinitely many flying discs. That is, if Δ_f illustrates that a closed braid X admits an elementary flype— $\beta(X) = W_1 U W_2 \sigma_{n-1}^{\pm 1}$ —then there are infinitely flying discs illustrating the same flype. Specifically, for

$W_1, W_2 \in \mathcal{W}^{n-2}$, $U \in \mathcal{U}^{n-1}$ and any word $\alpha \in \mathcal{W}^{n-3}$ we have an associated braid/flyping-disc pair $(X', \Delta_f(\alpha))$ with $\beta(X') = [W_1\alpha]U[\alpha^{-1}W_2]\sigma_{n-1}^{\pm 1}$. So for $X' \xrightarrow{N} X_{\mathcal{I}}'^{\eta}$ we have the family of resulting pairs $(X_{\mathcal{I}}'^{\eta}, \Delta_f'^{\eta}(\alpha))$.

Thus, we have an argument similar to the one for thin exchange moves. For a positive (respectively, negative) elementary flype we take a sufficiently complex α (say powers of full twists on $n - 2$ strands) to ‘grab’ all the strands that are common to the W_1 and W_2 braid blocks in the angular support of h_2 (respectively, h_3). The winding behavior of leaves in the radial foliation of $\Delta_f'^{\eta}(\alpha)$ can be used to create a product structure 3-ball $B^{1,2}$ (respectively, $B^{2,3}$) that intersects only these $n - 2$ strands and has h_2 ’s (respectively, h_3 ’s) angular support. Next, using \mathcal{O}_4 we can push out all vertical arcs in $B^{1,2}/B^{2,3}$. Finally, using $B^{1,2}/B^{2,3}$ we can reposition that the remaining braiding occurring in the angular support h_2/h_3 below all of the $n - 2$ strands intersecting $B^{1,2}/B^{2,3}$. This yields a $X_{\mathcal{I}}'^{\eta}$ such that for $X_{\mathcal{I}}'^{\eta} \xrightarrow{B} X'$ we have $\beta(X') = W_1'UW_2'\sigma_{n-1}^{\pm 1}$.

4 Algorithm.

We now describe how to utilize Theorem 4 to construct an algorithm for determining whether a give closed braid admits one of our specified braid isotopies, i.e. the algorithms of Corollary 5. In each case we start with a closed n -braid X and readily produce an arc presentation X^{η} . For each isotopy type the algorithm proceeds by first specifying a choice of shearing intervals \mathcal{I} .

4.1 Algorithm for destabilization isotopy.

For the destabilization isotopy we need only specify the location of one shearing interval. But, by the argument of Lemma 8, we can chose any angle θ . Let $[\theta - \epsilon, \theta + \epsilon] = \mathcal{I}$.

Now, we list the possibilities. For X be a link of k -components there are k possible destabilizations, where $k \leq n - 1$. So for a choice of component which has winding around \mathbf{A} greater than one we pick an edge-path $\mathcal{E} = h_1 \dot{\cup} v_1 \dot{\cup} h_2 \dot{\cup} \dots \dot{\cup} h_l$ such that: 1) $h_1 \cap \mathcal{I} \neq \emptyset \neq h_l^{\eta} \cap \mathcal{I}$; and, 2) $\mathcal{E} \cap \mathcal{I} = (h_1^{\eta} \cup h_l^{\eta}) \cap \mathcal{I}$.

We now apply our elementary moves to $X_{\mathcal{I}}'^{\eta}$ with the proviso that no shearing occur on the horizontal arcs of \mathcal{E} .

By Proposition 7 after a finite number of elementary moves either there are no more complexity reducing elementary moves to apply, or we have a resulting edge-path $\mathcal{E}' = h_1' \dot{\cup} v_1' \dot{\cup} h_2'$ such that: 1) $h_1' \cap \mathcal{I} \neq \emptyset \neq h_2' \cap \mathcal{I}$; and, 2) the horizontal positions of h_1' and h_2' are consecutive. If it is the latter then we are done. If it is the former then we start again with a new edge-path

on a different component of X . After a finite number of iterations this process will stop by either finding a destabilization or determining that X does not admit a destabilization.

4.2 Algorithm for thin exchange move.

For the exchange move isotopy we need only specify the location of two shearing interval which will necessarily have differing edge assignments. Again, by the argument of Lemma 8, we see that we in fact can chose $\mathcal{I} = \{[\theta_1 - \epsilon_1, \theta_1 + \epsilon_1], [\theta_2 - \epsilon_2, \theta_2 + \epsilon_2]\}$ where over the interval $[\theta_2, \theta_1]$ there are no vertical arcs. Specify the interval $[\theta_1 - \epsilon_1, \theta_1 + \epsilon_1]$ as a back edge and $[\theta_2 - \epsilon_2, \theta_2 + \epsilon_2]$ as a front edge.

We consider the possibilities. They include choices an edge-path $\mathcal{E} \subset X^\eta$ and an additional horizontal arc \hat{h} such that: 1) $\mathcal{E} = h_1 \dot{\cup} v_1 \dot{\cup} h_2 \dot{\cup} \dots \dot{\cup} h_l$; 2) h_1 has angular support containing $[\theta_2, \theta_1]$; 3) \mathcal{E} has angular support containing $[\theta_1, \theta_2]$. Obviously the number of possible choices for \hat{h} is bounded by the n . Then our choice of \mathcal{E} is bounded $n - 1$ and must satisfy the condition that $\hat{h} \cap \mathcal{E} = \emptyset$. With a choice of \mathcal{E} and \hat{h} in hand we proceed.

We now apply our elementary moves to $X_{\mathcal{I}}^\eta$ with the proviso that no shear horizontal exchange move occur on the horizontal portion of \mathcal{E} and on \hat{h} .

By Proposition 7 after a finite number of elementary moves either there are no more complexity reducing elementary moves to apply, or we have a resulting edge-path $\mathcal{E}' = h'_1$ such that the angular support of h'_1 contains the intervals $[\theta_1, \theta_2]$. Since the horizontal arc \hat{h} remains unaltered, the angular support of \hat{h} still contains the interval $[\theta_2, \theta_1]$. If it is the latter then we are done. (We then can use the horizontal arcs h'_1 and \hat{h} as the horizontal boundary of an obvious exchange disc.) If it is the former then we start again with a new choice of \mathcal{E} & \hat{h} pair. After a finite number of iterations this process will stop by either finding an exchange move or determining that X does not admit an exchange move.

To find thin exchange moves we need to repeat this procedure over all possible choices of shearing intervals, initial horizontal arcs and edgepaths. Finally, over all exchange moves that are recognized by an easy strand count we can pick the ones that are thin exchange moves.

4.3 Algorithm for an elementary flype.

For the flype isotopy we need only specify the location of three shearing interval. Again, by Lemma 8, we can choose \mathcal{I} to be $\{[\theta_1 - \epsilon_1, \theta_1 + \epsilon_1], [\theta_2 - \epsilon_2, \theta_2 + \epsilon_2], [\theta_3 - \epsilon_3, \theta_3 + \epsilon_3]\}$ where over $[\theta_1, \theta_2] \dot{\cup} [\theta_2, \theta_3]$ there are no vertical arcs of X^η .

We concern the final possibilities. They include choices an edge-path $\mathcal{E} \subset X^\eta$ and two additional horizontal arcs \hat{h}_1, \hat{h}_2 such that: 1) $\mathcal{E} = h_1 \dot{\cup} v_1 \dot{\cup} h_2 \dot{\cup} \dots \dot{\cup} h_l$; 2) \mathcal{E} has angular support

containing $[\theta_3\theta_1]$; 3) \hat{h}_1 has angular support containing $[\theta_1, \theta_2]$; and, 4) \hat{h}_2 has angular support containing $[\theta_2, \theta_3]$. Obviously the number of possible choices for \hat{h}_1 & \hat{h}_2 is bounded by the $\frac{n(n-1)}{2} - n$ chosen 2 ways. Our choice of \mathcal{E} is bounded by $n - 2$. With a choice of \mathcal{E} and \hat{h}_1, \hat{h}_2 in hand we continue.

We now apply our elementary moves to $X_{\mathcal{I}}^{\eta}$ with the proviso that no shear horizontal move occur on the horizontal portion of \mathcal{E} , \hat{h}_1^{η} , or \hat{h}_2^{η} . Moreover, we require that side on which shear horizontal exchange move or shear vertical simplification be consistent with the edge assignment of our shearing intervals as specified in §4.1 and 4.2.

By Proposition 7 after a finite number of elementary moves either there are no more complexity reducing elementary moves to apply, or we have a resulting edge-path being $\mathcal{E}' = h'_1 \dot{\cup} v' \dot{\cup} h''_1$ such that the angular support of \mathcal{E}' contains the interval $[\theta_3, \theta_1]$. Since the horizontal arc \hat{h} remains unaltered, the angular support of \hat{h}_1 & \hat{h}_2 still contain the appropriate intervals of \mathcal{I} . (We then can use the horizontal arcs $\mathcal{E}', \hat{h}_1, \hat{h}_2$ as the horizontal boundary of an obvious flying disc.) If it is the latter then we are done. If it is the former then we start again with a new choice of triple $\mathcal{E}, \hat{h}_1, \hat{h}_2$. After a finite number of iterations this process will stop by either finding an elementary flype or determining that X does not admit a flype. Reiterating this procedure over all possible choices (shearing intervals, initial horizontal arcs and edgepaths) we can recognize all possible braid preserving flypes.

4.4 Proof of Theorem 6

Since we now have an algorithm for when a closed braid admits a thin exchange move or elementary flype we take note of the finite possibilities for a closed braid X . Referring back to the “connect the dots” discussion prior to the statement of Theorem 4, the finiteness of $\mathcal{C}(X_{\mathcal{I}}^{\eta})$ tells us that there are only finitely many possible thin exchange moves or elementary flypes to perform. Performing a thin exchange move or elementary flype will result in a X' braid. We then use the classical solutions to the conjugacy problem to see whether X' and a given closed braid Y are braid isotopic. For thin exchange move we are only checking to see if Y is conjugate to the appropriate closed braid associated with $X' = W\tau_{[s,t+1]}U\tau_{[s,t+1]}^{-1}$, i.e. are X and Y related by a *single* thin exchange move. As an aside, the case where $Y = X$ is of particular interest. In a fashion similar to the parallel strands in a double destabilization, A. V. Mal'jutin's work points to the possibility that WU may be conjugate to $W\tau_{[s,t+1]}^k U\tau_{[s,t+1]}^{-k}$ for any integer value of k .

To determine whether X admits a double destabilization we first apply our exchange move algorithm to determine whether X admits a thin exchange move with $t - s = 2$ and $U \in \mathcal{U}^{n-3}$. If X admits such a thin exchange move then over all such thin exchange moves, we apply our destabilization algorithm to determine if the U block of $X = WU$ admits a destabilization.

If U admits a destabilization, we do the destabilization and applying our destabilization once more to the resulting U -block. If the resulting U -block also destabilizes then by inspection we determine whether U was conjugate to $\mathbf{X}^{\epsilon-2}$.

4.5 Destabilization example.

We now give an extended example illustrating how the algorithm is implemented on a closed braid that admits a destabilization. Figure 19 is a standard projection of a 4-braid, X , drawn on a portion of C_1 . (C_1 is represented as a rectangular disc with its left and right edges identified.) We will step through the implementation of the algorithm by first performing the transition $X \xrightarrow{N} X^\eta$ which gives us the arc presentation illustrated in Figure 20. The complexity measure of the arc presentation in Figure 20 is just the number of vertical arcs, i.e. 54.

The arc presentation in Figure 20 clearly has numerous places where the number of arcs can be reduced by the application of either vertical or horizontal simplifications. Without altering the number of crossings in the arc presentation we apply a number of such simplifications until we obtain the arc presentation illustrated in Figure 21. The complexity measure of X^η is now 13.

Next we perform, first a horizontal exchange move. The portion of the arc presentation where this horizontal exchange move occurs is indicated by a transparent heavy red edge-path that is a vertical-horizontal-vertical path. After the performance of this exchange move it is possible to perform two horizontal simplifications. The portion of the arc presentation involved in these simplification is indicated by a transparent heavy red edge-path that is a vertical-horizontal-vertical-horizontal path. These two horizontal simplifications reduced the complexity of X^η to 11. Finally, we perform two more horizontal exchange moves that can be sequenced independent of our previous operations. They are the two horizontal arc that are indicated by the heavy/transparent magenta horizontal arcs. These operations result in the arc presentation illustrated in Figure 22.

We now “extend” the right side of the rectangular disc so as to include a front edge shearing interval and we consider $X_{\mathcal{I}}^\eta$. The complexity of $X_{\mathcal{I}}^\eta$ is initially the same as that of X^η , namely 11. We now perform a shear horizontal exchange move and then a horizontal simplification using the edge-path in Figure 22 that is indicated by the heavy/transparent red path. This will yield the arc presentation illustrated in Figure 23. (This could also be seen as the performance of a shear vertical simplification.) The complexity of the resulting $X_{\mathcal{I}}^\eta$ is reduced by one to 10.

The alteration from Figure 23 to 24 is achieved by a shear vertical simplification applied to the vertical-horizontal red edge-path. The complexity measure of $X_{\mathcal{I}}^\eta$ is reduced by a count of one to 9.

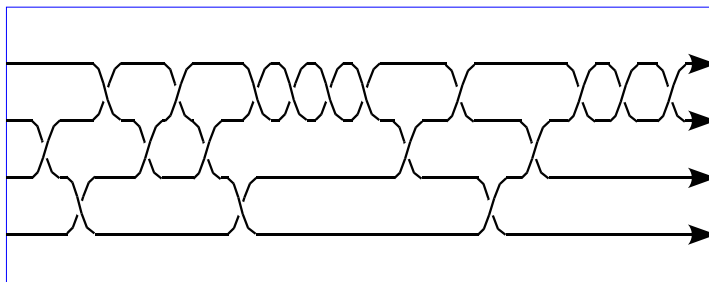


Figure 19: The rectangular disc is a portion of C_1 where we identify the left and the right edges. With this representation of C_1 we then have a standard projection of a 4-braid. We

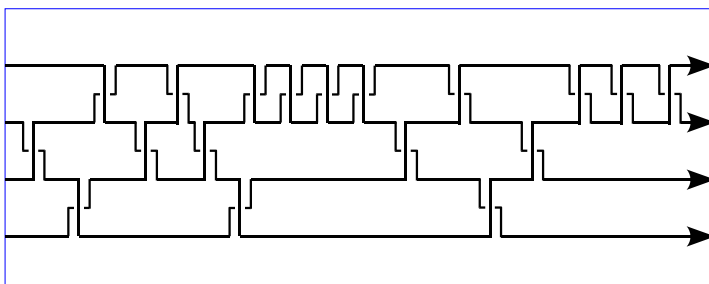


Figure 20: The arc presentation is directly obtained from the standard projection in Figure 19.

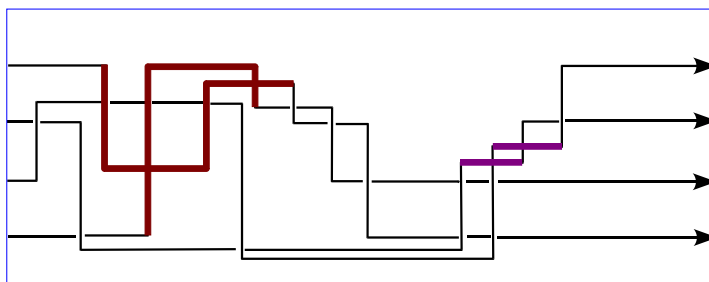


Figure 21: We indicate a vertical-horizontal-vertical (red) edge-path upon which we will perform a horizontal exchange move; a vertical-horizontal-vertical-horizontal (red) edge-path upon which we will perform two horizontal simplifications; and two (magenta) horizontal arcs upon which we will perform two more horizontal exchange moves.

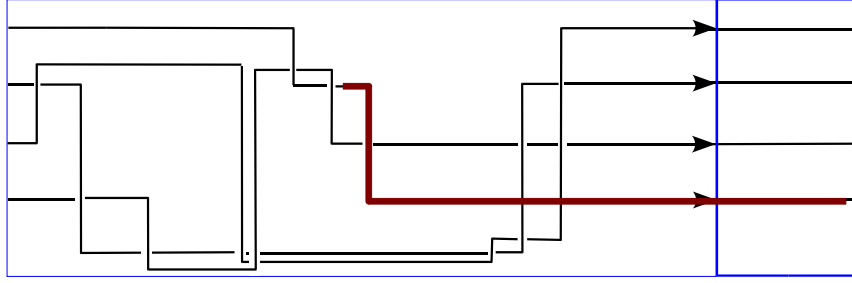


Figure 22: The right extended portion of the rectangular disc is our shearing interval. The red edge-path will be used in a shear horizontal exchange move.

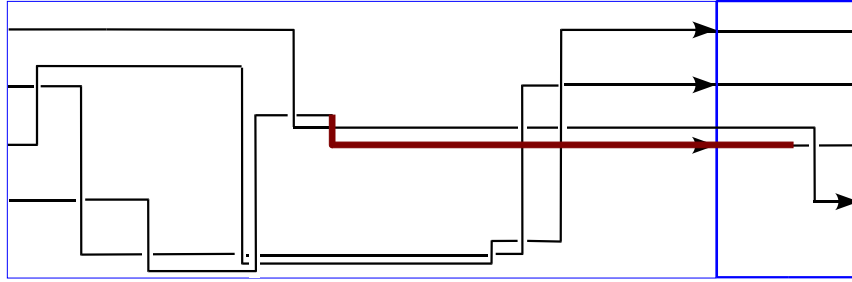


Figure 23: The red edge-path will be used in a shear vertical simplification.

On Figure 24 we perform two move shear horizontal exchange moves which will leave the complexity measure of $X_{\mathcal{I}}^{\eta}$ unchanged. The edge-paths where the shear horizontal exchange moves occur are indicated in heavy/transparent red again. The operation leaves unchanged the complexity of $X_{\mathcal{I}}^{\eta}$.

Finally, using the edge-path indicated by the red heavy/transparent path we perform one last shear horizontal exchange move to obtain the obvious destabilizing configuration illustrated in Figure 26. This last operation leaves the complexity of $X_{\mathcal{I}}^{\eta}$ unchanged.

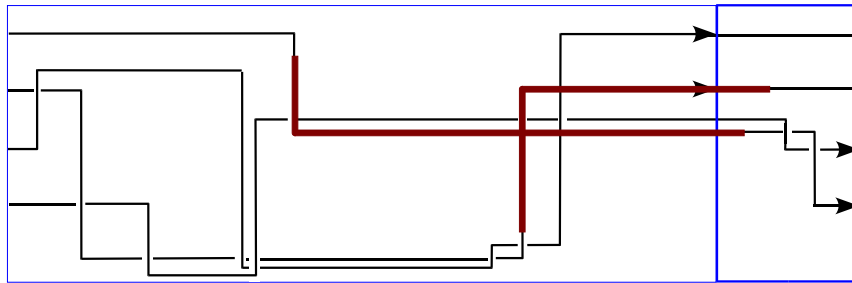


Figure 24: We indicate again by red edge-paths where we will perform two shear horizontal exchange moves.

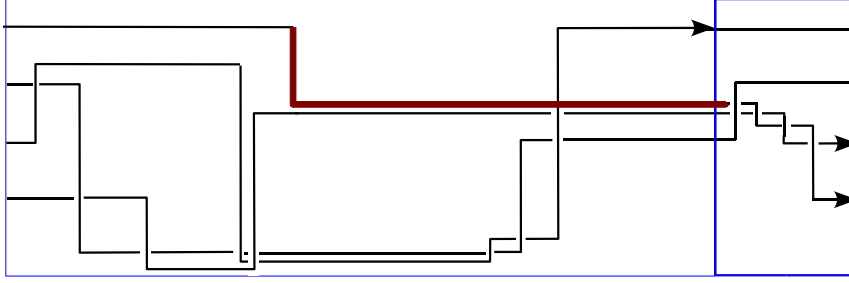


Figure 25: The red edge-path indicates where we will perform one last shear horizontal exchange move to achieve the obvious destabilizing arc presentation diagram in Figure 26.

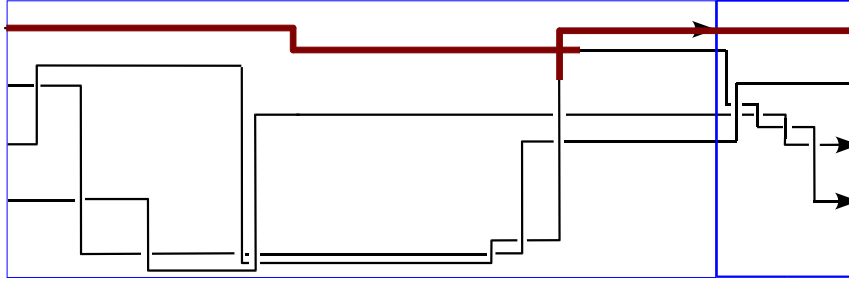


Figure 26: The red heavy/transparent edge-path picks out the configuration for an obvious destabilization.

4.6 Final remarks.

It is now useful to engage in some informed speculation on reasonable directions to push the machinery used to establish Theorem 4. The most reasonable direction is in adapting our machinery to recognize other isotopies of the MTWS calculus. Briefly, the statement of the MTWS says that of a given pair of positive integers (m, n) with $m \geq n$ there are only finitely many *templates*—isotopic pairs of block-strand diagram—which need be used to carry all closed m -braids to corresponding closed n -braids. The block-strand diagrams in Figure 4 illustrate the three simplest templates. In general, in order to recognize when a braid is carried by a specified block-strand diagram we need to account for both the blocks and the braiding of the strands. Adapting our Δ_ε^η -disc machinery we can clearly add more shearing intervals to our set \mathcal{I} . Since the angular support of any block can be made arbitrarily small, for a sufficient number of shearing intervals we can always place a Δ_ε^η in the complement of the braid the “jumps” from strand to strand as its boundary winds around the axis once. Moreover, we can allow the usage of multiple disjoint Δ_ε^η -discs so that their boundaries correspond to an unlink. Our machinery does not require that we have a Δ_ε^η -disc of a single component.

Finally, we do not have to restrict ourselves to just surfaces that are discs. Depending on

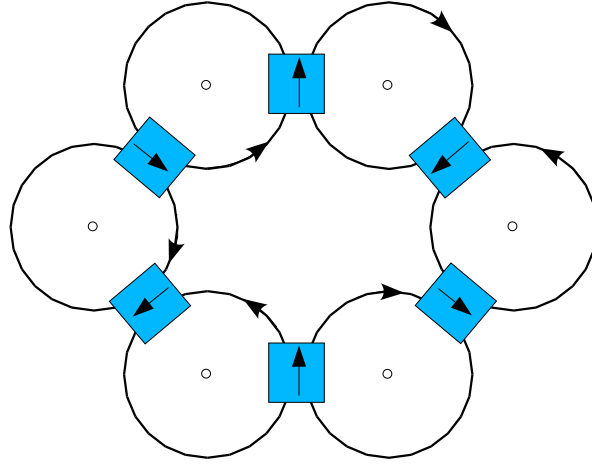


Figure 27: The 6 block can contain any braiding that is consistent with the direction of the indicate orientations. This block-strand diagram can be re-embedded in a braid fibration such that it is knotted. There will then exist a non-peripheral essential torus that will be a type $2k$ torus as described in [BM3]. We can use this torus to alter the braid presentation so as to recognize the admission of a cyclic move.

the configuration of the block-strand diagrams used in the isotopy template it may be more appropriate to use a closed surface in the complement of closed braid. For example, in [BM4] *cyclic moves* were analyzed. In Figure 27 we illustrate a block-strand diagram that has 6 blocks attached to 6 circles. This block-strand diagram can be embedded in braid fibration **H** such that it is in fact knotted. Thus, we can think of this block-strand diagram contained in a solid torus and we knot this diagram by knotting the solid torus. The embedding of the boundary of this solid torus will be a standard type $2k$ embedding as describe in [BM3]. We can then use the arc presentation machinery along with the coning discs to alter the arc presentation through horizontal and vertical exchange moves until our type $2k$ torus has a standard tiling.

Based upon this discussion there should be optimism about the possibility of recognizing any specified isotopy of the MTWS calculus. But, there is a cautionary point. Once it has been determined that a given closed braid admits a specified isotopy there may still be an ambiguity on how to apply the isotopy. For a closed braid that admits a destabilization, it is clear that one need only destabilize. But, for a closed braid that admits an exchange move, it is not clear which way to apply the exchange move to reduce. (See Figure 6 of [BM4] for an example of this pathology.) On this point more investigation is needed.

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